A TORSION THEORY FOR MODULES OVER RINGS WITHOUT IDENTITIES

By

JOHN MICHAEL KELLETT

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA
1968

To my wife

ACKNOWLEDGEMENTS

I would like to thank my chairman, Dr. W. Edwin Clark, for his instruction and guidance during my graduate study and the writing of this dissertation. The preparation of this paper has been a pleasant and rewarding experience due, in large measure, to Dr. Clark's patience, encouragement, and aid.

I am grateful to Dr. John Maxfield and Dr. Alex R. Bednarek for their timely encouragement and for their aid in procuring for me generous financial assistance, without which I would have been unable to continue my work.

To the members of my committee, to the Mathematics faculty, and especially to Dr. T. O. Moore and Dr. Y. L. Lee I would like to express my appreciation for their guidance and friendship.

Finally, I am deeply grateful to my wife, Christine, for her understanding, help, and patience during these years of study.

TABLE OF CONTENTS

INTRODUCTION

The initial objective of this dissertation was to consider the possibility of developing homological ring theory for a ring R without identity. As a possible generalization of the unital case, we consider the full subcategory ${}_R\mathfrak{R}$ of ${}_R\mathfrak{M}$, the category of all R-modules, consisting of those modules M such that R does not annihilate any nonzero element of M. We call the modules in ${}_R\mathfrak{R}$ torsion free. This category has the desired property that if R has an identity, then ${}_R\mathfrak{R}$ is the category of unital modules. The category ${}_R\mathfrak{R}$ is shown to be closed under taking submodules, products, and extensions and is the largest subcategory of ${}_R\mathfrak{M}$ which has these properties and contains no nonzero modules M such that RM = 0.

A basic theorem in homological ring theory is that if R has an identity, then the following are equivalent:

(1) All unital modules are projective;

(2) All unital modules are injective; and

(3) R is semi-simple Artinian.

One of our main results is the extension of the above theorem to rings without identities by replacing unital modules with torsion free modules.

The main difficulty encountered in working with ${}_R\mathfrak{K}$, as compared to the category of unital modules, is that ${}_R\mathfrak{K}$ is not, in general, closed under taking quotients. In fact, a basic result is that if the right annihilator of R is zero, then ${}_R\mathfrak{K}$ is closed under taking quotients if and only if R has an identity. We say that a submodule N of an R-module M is <u>closed</u> if M/N is in ${}_R\mathfrak{K}$. As with normal subgroups in the category of groups and ideals in the category of rings, closed submodules are of particular importance. We show that the closed submodules of an R-module form a complete modular lattice and give necessary and sufficient conditions for R such that each such lattice is complemented.

${}_R\mathfrak{K}$ is shown to be a torsion free class for a torsion theory (in the sense of S. E. Dickson [2]) on ${}_R\mathfrak{M}$. A. W. Goldie in [5] and J. P. Jans in [8] have considered other torsion theories on ${}_R\mathfrak{M}$, and we have found necessary and sufficient conditions such that their torsion theories coincide with the torsion theory determined by ${}_R\mathfrak{K}$ for rings with zero right annihilator ideals.

SECTION I

THE CLASSES ${}_R\mathfrak{M}$, ${}_R\mathfrak{T}$, AND ${}_R\mathfrak{F}$ (R-MODULES, TORSION R-MODULES, AND TORSION FREE R-MODULES)

In this section we define and discuss the elementary properties of torsion modules and torsion free modules. We prove the expected result that every module contains a unique maximal torsion submodule whose quotient is torsion free. It turns out that if the right annihilator of a ring R is 0, then R has an identity if and only if the class ${}_R\mathfrak{F}$ of torsion free R-modules is closed under homomorphic images. A description of free modules relative to ${}_R\mathfrak{F}$ is given that is very useful in the later sections.

1.1 Definition. We shall call an R-module M torsion free if, for each nonzero element m in M, $Rm \neq 0$.

Of course, if R is the ring of integers and M is a unital R-module, then the above definition differs from the usual notion of torsion freeness. See Section II for a justification of this terminology.

Notation. ${}_R\mathfrak{F}$ will be used to designate the category of torsion free R-modules. It is a full subcategory of ${}_R\mathfrak{M}$.

1.2 Proposition. ${}_R\mathfrak{R}$ is closed under taking submodules.

Proof. The proof is clear from the definition.

1.3 Proposition. If R has an identity, then ${}_R\mathfrak{R}$ is the category of unital modules.

Proof. The category of unital modules is clearly a subcategory of ${}_R\mathfrak{R}$.

Let $M \in {}_R\mathfrak{R}$ and $0 \neq m \in M$. Then $r(1m - m) = r1(1m - m) = 0$ for all $r \in R$. Hence $R(1m - m) = 0$, which implies that $1m = m$, where 1 is the identity of R. Therefore M is a unital module.

1.4 Proposition. ${}_R\mathfrak{R}$ is closed under taking factors if and only if for each torsion free module M and each $m \in M$, $m \in Rm$.

Proof. Assume ${}_R\mathfrak{R}$ is closed under taking factors.

Let $M \in {}_R\mathfrak{R}$ with $m \in M$; then M/Rm is torsion free by assumption.

Hence $m \in Rm$ since $R(m + Rm) = Rm = 0$.

Conversely, assume for each torsion free module M and each $m \in M$, $m \in Rm$.

Let L be a submodule of the torsion free module M. If $R(m + L) = L$, then $Rm \leq L$ and $m \in L$ by assumption. Therefore M/L is torsion free.

1.5 Proposition. If R is nilpotent, then ${}_R\mathfrak{R}$ contains only the zero module.

Proof. Assume $R^k = 0$ for k, a positive integer. Note that if M is a nonzero torsion free module, then RM is a nonzero torsion free module.

Inductively, R^kM is a nonzero torsion free module if M is nonzero torsion free. However, $R^kM = 0$ for all R-modules since $R^k = 0$. Therefore ${}_R\mathfrak{R}$ contains only the zero module.

Definition. Let $R^1 = R \times Z$(Z the ring of integers) where addition is defined by $(r,k) + (r',k') = (r + r', k + k')$ and multiplication by $(r,k)(r',k') = (rr' + kr' + k'r, kk')$. As usual, we identify R with R $\{0\}$; then R^1 is considered an R-module where the module multiplication is defined in terms of the ring multiplication.

1.6 Proposition. R^1 is a torsion free R-module if and only if for each nonzero element $(r,-n)$ of R^1 there exists $a \in R$ such that $ar \neq na$.

Proof. The proof is immediate from the definition.

1.7 Proposition. If R is a ring with a right identity e, then $I = \{r \in R | er = 0\}$ is an ideal of R and R/I has the identity $(e + I)$.

Note. $I = r(e) = (1 - e)R$ where for $x \in R$, $r(x) = \{y \in R | xy = 0\}$ and $(1 - e)R = \{r - er | r \in R\}$.

1.8 Proposition. If R is a ring with a right identity e and $I = r(e)$, then an R-module M is torsion free if and only if M is a unital R/I module (under the definition $(r + I)m = rm$).

Proof. If M is a unital R/I module, then M is clearly a torsion free R-module.

If M is a torsion free R-module, then for $m \in M$ and $r \in I$, $Rrm = (Re)rm = R(er)m = 0$. This implies $rm = 0$, since M is torsion free. Hence $IM = 0$. Therefore M can be considered an R/I module. It follows, from Proposition 1.3, that M is a unital R/I module.

Note that if e is a right identity for R, then $r(e) = r(R) = \{x \in R \mid Rx = 0\}$. Hence we have

1.9 Proposition. If R is a torsion free R-module and R has a right identity, then R has an identity.

1.10 Definition. A submodule L of an R-module M will be said to be closed in M if M/L is torsion free.

1.11 Proposition. L, a submodule of M, is closed in M if and only if for each $m \in M$, $Rm \subseteq L \Rightarrow m \in L$.

Proof. Immediate from the definition.

1.12 Proposition. The intersection of closed submodules of M is closed in M.

Proof. This proposition follows from 1.11.

1.13 Proposition. If L is a closed submodule in N and N is a closed submodule in M, then L is a closed submodule in M.

Proof. Let $m \in M$ and $Rm \subseteq L$; then Rm is contained in N. By (1.11) $m \in N$, which implies $m \in L$.

1.14 Proposition. If L is closed in M and H is a submodule of M, then $L \cap H$ is closed in H.

Notation. For an R-module N, let $T_R(N)$ be the intersection of all closed submodules of N. $T_R(N)$ will be written $T(N)$ when there is no possibility of confusion.

1.15 Definition. An R-module K will be called a torsion module if K is the only closed submodule of K. That is, K is torsion if its only torsion free homomorphic image is 0.

Example. Let $R = 2Z$ (Z the integers) and L_1, L_2, L_3, and L_4 be the submodules of ${}_R R$, 4Z, 6Z, 8Z, and 12Z respectively. It is easily seen that R/L_1 and R/L_3 are torsion modules and R/L_2 is torsion free. R/L_4 is not torsion free since $R(6 - L_4) = 0$ and it is not torsion since R/L_2 is a homomorphic image of R/L_4, which implies R/L_4 contains a proper closed submodule.

1.16 Proposition. If N is an R-module, then N/T(N) is a torsion free module.

Proof. The proof follows directly from Proposition 1.12 and the definition of T(N).

1.17 Proposition. If N is an R-module, then T(N) is a torsion module.

Proof. If H is closed in T(N), then H is closed in N since T(N) is closed in N. Therefore H closed in T(N) implies that H = T(N), since T(N) is the intersection of all closed submodules of N.

1.18 Proposition. If L is a torsion module and L is a submodule of N, then L is a submodule of T(N).

Proof. Assume L is a submodule of N. Then $T(N) \cap L$ is a closed submodule in L since T(N) is closed in N. Hence $T(N) \cap L = L$ and so $L \subset T(N)$.

<u>Comment</u>. From 1.17 and 1.18 we note that $T(M)$ is the union of all torsion submodules of M for any R-module M.

1.19 <u>Proposition</u>. (1) M is a torsion free module if and only if $T(M) = \{0\}$. (2) M is a torsion module if and only if $T(M) = M$.

<u>Proof</u>. If M is torsion free, then $\{0\}$ is a closed submodule. Hence $T(M) = \{0\}$. If $T(M) = \{0\}$, then $M/\{0\} \cong M$ is torsion free by 1.16. Part (2) follows from 1.17 and 1.18.

1.20 <u>Proposition</u>. If N is an R-module and M is a torsion free R-module, with g a homomorphism from N into M, then $N/\ker g$ is a torsion free module.

1.21 <u>Definition</u>. A left ideal K of R will be said to be <u>closed</u> if it is closed as a submodule of ${}_R R$.

<u>Notation</u>. $T_R(R)$ is the intersection of all closed left ideals of R.

1.22 <u>Proposition</u>. If M is a torsion free module, then $T(R)M = 0$.

<u>Proof</u>. Let $m \in M$ and $r \in R$ such that $rm \neq 0$. It is sufficient to show that $r \notin T(R)$. Define a mapping g from R into M by $g(r') = r'm$ for $r' \in R$. Clearly g is an R-module homomorphism, and r is not an element of the kernel of g since $rm \neq 0$.

By Proposition 1.20, $R/\ker g$ is torsion free. Therefore $\ker g$ is a closed submodule of R which implies $T(R)$ is a submodule of $\ker g$. Hence r is not an element of $T(R)$.

<u>Comment</u>. From 1.22 we have that $T(R)M \subseteq T(M)$.

1.23 Proposition. $T(R)$ is an ideal of R. (This is noted by Jans in [8] in a more general context.)

Proof. $T(R)$ is a left ideal since it is a submodule of R. $R/T(R)$ is a torsion free module by 1.16. Hence $T(R)(R/T(R)) = 0$ by 1.22. Therefore $T(R)$ is a right ideal.

1.24 Proposition. If M is a torsion free module, then M is a $R/T(R)$ torsion free module.

Proof. This follows from 1.22 and 1.23.

1.25 Definition. We say that a ring is torsion free if ${}_R R$ is torsion free, i.e., if $T(R) = 0$.

1.26 Definition. Let M be an R-module and define:

$$M_1 = \{ m \in M \mid Rm = 0 \}$$

For α any ordinal number with the predecessor $\alpha - 1$ define:

$$M_\alpha = \{m \in M \mid Rm \subseteq M_{\alpha-1} \}$$

For α a limit ordinal define:

$$M_\alpha = \bigcup_{\beta<\alpha} M_\beta$$

1.27 Proposition. (a) M_α is a submodule of M for any ordinal α.

(b) M_α is a submodule of M_β if α is less than or equal to β.

(c) There exists an ordinal number σ such that $M_\alpha = M_\sigma$ for all α greater than or equal to σ.

(d) If $M_\sigma = M_{\sigma+1}$, then $M_\sigma = T(M)$.

Proof. (a) and (b) follow easily from the definition.

To prove (c), it is clear from (b) that there exists an ordinal number β such that $M_\beta = M_{\beta+1}$. Let σ be the least ordinal such that $M_\sigma = M_{\sigma+1}$. Assume that $M_\sigma = M_\sigma$ for all $\sigma < \alpha < \gamma$. If γ is a limit ordinal, then by the definition of M_γ it is clear that $M_\gamma = M_\sigma$. If γ has a predecessor $\gamma - 1$, then $M_\gamma = \{m | Rm \subseteq M_{\gamma-1}\} = \{m | Rm \subseteq M_\sigma\} = M_{\sigma+1} = M_\sigma$. Hence $M_\alpha = M_\sigma$ for all $\alpha \geq \sigma$ where σ is the least ordinal such that $M_\sigma = M_{\sigma+1}$. This proves part (c).

To prove (d) assume $M_\sigma = M_{\sigma+1}$. Let $m \in M$ and $Rm \subseteq M_\sigma$, then $m \in M_{\sigma+1}$ since $M_{\sigma+1} = \{m | Rm \subseteq M_\sigma\}$. Hence M_σ is a closed submodule of M. This implies T(M) is contained in M_σ. From the definition of torsion module, it is easily seen that M_1 is a torsion module; hence M_1 is contained in T(M). Assume M_σ is contained in T(M) for all $\alpha < \beta$. If β is a limit ordinal, then clearly M_β is contained in T(M). If β has a predecessor $\beta - 1$, then $m \in M_\beta$ implies Rm is contained in $M_{\beta-1}$, which is contained in R(M). Since T(M) is closed in M, $Rm \subseteq T(M)$, which implies $m \in T(M)$. This proves M_β is contained in T(M) for all ordinals β. Therefore if $M_\sigma = M_{\sigma+1}$, then $M_\sigma = T(M)$.

1.28 Proposition. If M is an R-module and $0 \neq m \in Rm$, then $m \notin T(M)$.

Proof. The proof follows from 1.27 if we note that $m \notin M_\alpha$ for any α by transfinite induction.

1.29 Proposition. If M is an R-module and there exist s, $r \in R$, and $m \in M$ such that $srm = rm \neq 0$, then $m \notin T(M)$.

Proof. By 1.28, $rm \notin T(M)$. Hence $m \notin T(M)$ since $T(M)$ is a submodule.

Notation. For any ordinal number α define $R_\alpha = ({}_R R)_\alpha$ as in Definition 1.26.

1.30 Proposition. For any ring R,

(a) R_α is an ideal of R for any ordinal α.

(b) $R_\alpha \subset R_\beta$ if and only if $\alpha \leq \beta$.

(c) If $R_\alpha = R_{\alpha+1}$, then $R_\alpha = T(R)$.

Proof. (b) and (c) are immediate from 1.27 and (a) is easily verified.

Example. Let A be the set of all ordinal numbers less than or equal to a given ordinal α_0 and V be a vector space with a basis $\{v_\beta\}_{\beta \in A}$. Let R be the ring of all linear transformations γ on V of finite rank such that for any $\beta \in A$, $\gamma(v_\beta) \neq 0$ implies that $\gamma(v_\beta) \in [v_\sigma]_{\sigma<\beta}$. It is easily seen that R_ρ is properly contained in R if ρ is less than α_0 and $R = R_\rho$ if ρ is greater than α_0, where R_ρ is as in 1.26. Hence for any ordinal σ there exists a torsion module M with M_σ properly contained in M.

1.31 Proposition. (a) If R has a left identity, then $T(R) = 0$.

(b) If R has a right identity or, more generally, if $R^2 = R$, then $R_1 = T(R)$.

Recall that I is a nil ideal of a ring R if I is an ideal of R and for each $x \in I$, there exists n such that $x^n = 0$.

The <u>nil radical</u> of a ring R is the union of all nil ideals of R. The nil radical is a nil ideal of R (see, e.g., [3, p. 19]).

1.32 <u>Proposition</u>. The nil radical N of R is closed in R, i.e., it is closed as a left ideal of R.

<u>Proof</u>. Suppose $r \in R$ and $Rr \subseteq N$. Note $(RrR + Rr + rR + Zr)^3$ is contained in RrR. Since N is an ideal, RrR is contained in N which implies $(RrR + Rr + rR + Zr)^3$ is contained in N. It follows that $(RrR + Rr + rR + Zn)$ is a nil ideal. Therefore $(RrR + Rr + rR + Zr)$ is contained in N and $r \in N$. By Proposition 1.11, N is a closed left ideal.

1.33 <u>Corollary</u>. T(R) is contained in the nil radical of R.

<u>Example</u>. Let $R = \prod_{k \in N} R_k$ where $R_k = (2)/(2^k)$.

It is easily checked that ${}_R R_k$ is a torsion R-module for $k \in N$. However, ${}_R R$ is not a torsion module since it is not a nil ring. Hence the direct product of torsion R-modules is not necessarily a torsion module. Clearly we could obtain similar examples by replacing R_k by any nilpotent ring whose index of nilpotency is k.

Recall that an element x of R is called <u>left quasi-regular</u> if there exists y in R such that $x + y + yx = 0$. If I is a left ideal of a ring R and if every element in I is left quasi-regular, then I is called a <u>left quasi-regular left ideal</u>. Let J be the union of all left quasi-regular left ideals of a ring R, then J is a left quasi-regular

two-sided ideal. J is called the <u>Jacobson Radical</u> of the ring R, (see, e.g., [3, p. 92]).

1.34 <u>Proposition</u>. J, the Jacobson Radical of R, is closed in R.

<u>Proof</u>. As is well known, the nil radical of R is contained in the Jacobson Radical of R ([3, p. 93]). Hence 0 is the nil radical of R/J. By 1.32, 0 is a closed left ideal of R/J, i.e. R/J is a torsion free R/J-module. It is easily seen that this implies R/J is a torsion free R-module.

1.35 <u>Proposition</u>. If R is an ideal of S and S has an identity, then $T_R({}_RS) = \{s \in S | Rs \subset T_R({}_RR)\}$.

<u>Proof</u>. In this proof T always means T_R. Let $L = \{s \in S | Rs \subset T({}_RR)\}$. By 1.16, we have $T(R)(S/T(S)) = 0$, i.e., $T(R)S \subseteq T(S)$. Since S has an identity element, $T(R) \subseteq R(S)$. For $s \in S$ and Rs a subset of T(R), we note that $s \in T(S)$ by Proposition 1.11. Hence L is contained in $T({}_RS)$.

Let $s \in S \setminus L$. Then there exists r such that $rs \notin T(R)$. This implies that Rrs is not a subset of T(R) by Proposition 1.11 since T(R) is closed in R. It follows that rs is not an element of L, which implies that L is closed in S. Hence $T({}_RS)$ is contained in L.

Therefore $T(S) = L$.

1.36 <u>Corollary</u>. $T_R(R^1) = \{(r,k) | sr + ks \in T_R(R)$, for all $s \in R\}$.

1.37 <u>Corollary</u>. If $T(R) = 0$, then $T(R^1) = \{(r,k) | sr = -ks$ for all $s \in R\}$.

1.38 <u>Proposition</u>. $r \in T(R)$ if and only if $rM = 0$ for all torsion free modules M.

<u>Proof</u>. If $r \in T(R)$, then $rM = 0$ for all torsion free modules by Proposition 1.22.

Assume $r \in R \backslash T(R)$. Note that $r(0,1) + T(R^1)) = (r,0) + T(R')$ and $Rr \subseteq T(R)$; hence $(r,0) \notin T(R^1)$ by 1.36. Therefore $r((0,1) + T(R^1)) \neq 0$, i.e., $r(R^1/T(R^1)) \neq 0$.

<u>Comment</u>. It is obvious from the above argument that $((0,1) + T(R')) \neq 0$.

1.39 <u>Proposition</u>. If R is torsion free, then the following are equivalent:

(1) R has an identity.

(2) ${}_R\mathfrak{R}$ is closed under taking factors.

<u>Proof</u>. If R has an identity, then (2) follows from Proposition 1.3.

Conversely, we assume ${}_R\mathfrak{R}$ is closed under taking factors. By Proposition 1.4, we note that $(0,1) + T(R^1) \in R((0,1) + T(R^1))$. Hence $(0,1) + T(R^1) = (r,0) + T(R^1)$ for some $r \in R$. This implies that $(r,-1) \in T(R^1)$ for some $r \in R$. Therefore, by Proposition 1.37, $sr = s$ for all $s \in R$, i.e., R has a right identity. It follows from Proposition 1.9, that R has an identity since R has a right identity.

1.40 <u>Corollary</u>. The following are equivalent:

(1) $R/T(R)$ has an identity.

(2) ${}_R\mathfrak{R}$ is closed under taking factors.

Proof. Note that since $T(R)M = 0$ for all $M \in {}_R\mathfrak{R}$ it is clear that the category ${}_R\mathfrak{R}$ is isomorphic to the category ${}_{R/T(R)}\mathfrak{R}$.

1.41 Proposition. If the additive group R^+ of R is divisible and if $T(R) = 0$, then $T({}_RR^1) \neq 0$ if and only if R has an identity.

Proof. Assume that R^+ is divisible, that R is a torsion free R-module, and that $T(R^1) \neq 0$. Then there exists $b \in R$ and $k \in Z$ such that $(b,k) \neq 0$ and $rb = kr$ for all $r \in R$ by Proposition 1.37. Note that if $b \neq 0$, then $k \neq 0$ since $Rb \neq 0$, and $k \neq 0$ implies $b \neq 0$ since $kR = R$. It follows then that $b \neq 0 \neq k$, and $b = kc$ for some $c \in R$, since $R = kR$.

Now let $r \in R$ and let $r = kr'$. Then $rc = kr'c = r'kc = r'b = kr' = r$. Hence c is a right identity for R and, since R is torsion free, R has an identity by Proposition 1.9. Conversely, we note if R has an identity, e, then $R(e,-1) = 0$, which implies $(e,-1) \in T(R')$. This proves the proposition.

1.42 Proposition. If M is an R-module, N is a torsion free module, and g a homomorphism from M into N, then g can be factored through $M/T(M)$, i.e., there exists a unique homomorphism h from $M/T(M)$ into N such that $h \circ f = g$ where f is the canonical homomorphism onto $M/T(M)$.

Proof. Let g be a homomorphism from M into N, where N is torsion free. Then $M/\ker g$ is torsion free. It follows that $T(M)$ is contained in $\ker g$. Therefore

g can be factored through $M/T(M)$.

1.43 Proposition. The direct product of modules is torsion free if and only if the component modules are torsion free.

Note. It is easily seen that H is an R-module if and only if H is a (unital) R^1-module. For H an R-module, $(r,k) \in R^1$, and $h \in H$, $(r,k)h$ is defined to equal $rh + kh$. If H is an R^1-module, then rh is defined to equal $(r,0)h$ for $r \in R$ and $h \in H$. It is also easy to see that the map, α, is an R homomorphism from ${}_RA$ into ${}_RB$ if and only if α is an R^1 homomorphism from ${}_{R^1}A$ into ${}_{R^1}B$. Thus, the category of all R-modules is isomorphic to the category of all unital R^1-modules.

1.44 Proposition. $\oplus \sum_{i \in I} M_i / T\left(\oplus \sum_{i \in I} M_i\right)$ is isomorphic to $\oplus \sum_{i \in I} M_i/T(M_i)$.

Proof. Since, in general, if N_i is a submodule of M_i, we have $(\oplus_i M_i)/(\oplus N_i) \cong \oplus_i (M_i/N_i)$, it suffices to prove that $T(\oplus_i M_i) = \oplus_i T(M_i)$. (For simplicity assume all direct sums are internal.) Since a direct sum of torsion free modules is torsion free, $\oplus_i (M_i/T(M_i)) \cong (\oplus_i M_i/(\oplus_i T(M_i))$ is torsion free. Hence $\oplus_i T(M_i)$ is closed and therefore contains $T(\oplus_i M_i)$. On the other hand, each $T(M_i)$ is torsion and a submodule of $\oplus M_i$, and so each $T(M_i)$ is contained in $T(\oplus_i M_i)$. It follows that $\oplus_i T(M_i) \subset T(\oplus_i M_i)$.

1.45 Definition. We call F an $\mathfrak{R}$-free module if $F \in {}_R\mathfrak{R}$ and F has a generating set, A, such that any mapping from A to a torsion free module, M, can be extended uniquely to a homomorphism from F into M. A is said to be a free basis for F in the category ${}_R\mathfrak{R}$.

1.46 Proposition. Let F be a free R^1-module with basis A. Then ${}_RF/T({}_RF)$ is an $\mathfrak{R}$-free R-module with basis $\{a + T({}_RF) \mid a \in A\}$.

Proof. Let $\pi : F \to F/T(R)$ be the natural projection. Let $g : \pi(A) \to M$ where M is any torsion free R-module. Then $g\pi$ yields a mapping from A to M which has a unique extension $f : F \to M$. Here we consider M as an R^1-module and so f is an R^1-homomorphism and hence also an R-homomorphism. By 1.20, $T(F) \subset \ker f$ and so f induces a unique mapping g^*, making the following diagram commutative.

$$\begin{tikzcd} F \arrow[r, "\pi"] \arrow[dr] & F/T(F) \arrow[d, "g^*"] \\ & M \end{tikzcd}$$

Then for $a \in A$, $g^*(\pi(a)) = f(a) = f(\pi(a))$. That g^* is unique follows from the fact that $\pi(A)$ generates $F/T(F)$.

1.47 Proposition. An r-module F is $\mathfrak{R}$-free if and only if it is isomorphic to a direct sum of copies of ${}_RR^1/T({}_RR^1)$.

Proof. It is immediate from the definition that $\mathfrak{R}$-free modules whose bases have the same cardinality are isomorphic. Hence, by 1.46, each $\mathfrak{R}$-free module is isomorphic to $F/T(F)$ where F is a free R'-module. Since F is

a direct sum of copies of R^1, we deduce immediately from 1.44 that $F/T(F)$ is a direct sum of copies of ${}_R R^1/T({}_R R^1)$.

1.48 Definition. A torsion free R-module, P, will be said to be $\mathfrak{R}$-projective if for A and B, any torsion free modules, with f a homomorphism from A onto B and g a homomorphism from P into B, there exists h, a homomorphism from P into A, such that $f \circ h = g$.

The reader should note that this definition differs from the usual definition of a projective object in a category since, as we shall soon show, epimorphisms in the category ${}_R\mathfrak{R}$ need not be onto.

A torsion free R-module Q, is said to be $\mathfrak{R}$-injective if for C and D, any torsion free modules with i a monomorphism from C into D and k a homomorphism from C into Q there exists g, a homomorphism from D into Q such that $g \circ i = k$.

As usual, one shows:

1.49 Proposition. An $\mathfrak{R}$-free module is $\mathfrak{R}$-projective.

1.50 Proposition. Every torsion free module is a homomorphic image of an $\mathfrak{R}$-projective module.

1.51 Proposition. Every $\mathfrak{R}$-projective module is a direct summand of an $\mathfrak{R}$-free module.

1.52 Definition. Recall that a morphism f in a category $\mathfrak{P}$ is said to be an epimorphism if whenever g and h are morphisms in $\mathfrak{P}$ such that $gf = hf$, then $g = h$.

1.53 Proposition. All epimorphisms in ${}_R\mathfrak{R}$ are onto if and only if $R/T(R)$ has an identity.

Proof. Assume all epimorphisms in ${}_R\mathfrak{R}$ are onto and suppose R/T(R) does not have an identity. By 1.40, there exists $M \in {}_R\mathfrak{R}$ with L a submodule of ${}_R\mathfrak{R}$ such that $M/L \notin {}_R\mathfrak{R}$. Let H equal the intersection of all closed submodules in M containing L (M is such a submodule) and note H is closed in M. Let f be the inclusion homomorphism from L into H and suppose gf = hf where g and h are morphisms in ${}_R\mathfrak{R}$. It follows that the ker(g-h) is a closed submodule of H containing L. Hence ker(g-h) = H since H by construction contains no proper closed submodules which contain L. This implies that f is an epimorphism which contradicts the assumption. Therefore R/T(R) has an identity.

The converse follows easily from 1.40.

Comment. It follows from 1.53 that if R does not have an identity and T(R) = 0, then there exists an $\mathfrak{R}$-free module which is not projective in the category ${}_R\mathfrak{R}$ in the categorical sense of projectivity. (See [10, p. 69].) Hence, if T(R) = 0 and all torsion free modules are assumed projective in ${}_R\mathfrak{R}$, then R has an identity.

SECTION II

THE COUPLE $({}_R\mathfrak{T}, {}_R\mathfrak{F})$ AS A TORSION THEORY AND THE CLASS OF TORSION RINGS

In this section it is shown that the couple $({}_R\mathfrak{T}, {}_R\mathfrak{F})$ satisfies a set of axioms given by S. E. Dickson for torsion theories in Abelian categories. It is proven that ${}_R\mathfrak{T}$ is the smallest torsion class for a torsion theory on ${}_R\mathfrak{M}$ which contains the class of R-modules M such that RM = 0. The torsion theory $({}_R\mathfrak{T}, {}_R\mathfrak{F})$ is shown to coincide with torsion theories of A. W. Goldie and J. P. Jans if R satisfies certain conditions. The properties of the class of rings such that ${}_R\mathfrak{F}$ is trivial, (i.e., that it only contains the zero module) are discussed.

2.1 <u>Definition</u> (S. E. Dickson). A <u>torsion theory</u> for ${}_R\mathfrak{M}$ consists of a couple $(\mathfrak{T}, \mathfrak{F})$ of classes of modules of ${}_R\mathfrak{M}$ satisfying the following axioms. $\mathfrak{T}$ is called the <u>torsion class</u> and $\mathfrak{F}$ the <u>torsion free</u> class with $M \in \mathfrak{T}$ called <u>$\mathfrak{T}$-torsion</u> and $F \in \mathfrak{F}$ called <u>$\mathfrak{F}$-torsion free</u>.

(1) $\mathfrak{T}$ and $\mathfrak{F}$ have only zero in common.

(2) If $T \in \mathfrak{T}$ and $T \to A \to 0$ is exact, then $A \in \mathfrak{T}$.

(3) If $F \in \mathfrak{F}$ and $0 \to H \to F$ is exact, then $H \in \mathfrak{F}$.

(4) For each M in ${}_R\mathfrak{M}$, there is an exact sequence $0 \to T \to X \to F \to 0$ with $T \in \mathfrak{T}$ and $F \in \mathfrak{F}$.

2.2 Definition (S. E. Dickson). A torsion theory $(\mathfrak{T},\mathfrak{F})$ is said to be closed under taking submodules if $\mathfrak{T}$ is closed under taking submodules.

Recall that a class of modules, $\mathfrak{D}$, of the category ${}_R\mathfrak{M}$ is said to be closed under taking extensions if $0 \to A \to B \to C \to 0$ is an exact sequence with A and C elements of $\mathfrak{D}$ implies B is an element of $\mathfrak{D}$.

2.3 The following are results proven by S. E. Dickson. In the statements, $\mathfrak{T}$ will designate a torsion class and $\mathfrak{F}$ a torsion free class for an arbitrary torsion theory on ${}_R\mathfrak{M}$.

(1) $\mathfrak{F}$ is closed under direct products and $\mathfrak{T}$ is closed under direct sums.

(2) Both $\mathfrak{T}$ and $\mathfrak{F}$ are closed under extensions.

(3) $\mathfrak{T}$ is closed under taking subobjects if and only if $\mathfrak{F}$ is closed under taking minimal injectives.

(4) Given $\mathfrak{T}$ closed under taking factors, extensions, and direct sums, there exists a unique $\mathfrak{F} = \{F | \mathrm{Hom}(T,F) = 0,$ for all $T \in \mathfrak{T}\}$ such that $(\mathfrak{T},\mathfrak{F})$ is a torsion theory.

(5) Given $\mathfrak{F}$ closed under taking subobjects, extensions, and direct products, there exists a unique class

$$\mathfrak{T} = \{T | \mathrm{Hom}(T,F) = 0, \text{ for all } F \in \mathfrak{F}\}$$

such that $(\mathfrak{T},\mathfrak{F})$ is a torsion theory.

(6) For $M \in {}_R\mathfrak{M}$ there exists a unique largest submodule M_t of M which is a member of $\mathfrak{T}$ and $M/M_t \in F$. The T of axiom (4) equals this unique submodule

$$M_t = \ T \subseteq M | T \in \mathfrak{T}\} \text{ or } M_t = \cap\{S \subseteq M | M/S \in \mathfrak{F}\}.$$

(7) The correspondence $M \to M_t$ defines a functor $t : {}_R\mathfrak{M} \to {}_R\mathfrak{M}$ having the properties:

(a) Given $f : A \to B$, $t(f) : A_t \to B_t$ is the restriction of f,

(b) $t(A/A_t) = 0$,

(c) $t^2 = t$.

2.4 Proposition. $({}_R\mathfrak{T}, {}_R\mathfrak{R})$ is a torsion theory where ${}_R\mathfrak{T}$, ${}_R\mathfrak{R}$ are the class of torsion and torsion free modules as defined in 1.15 and 1.1.

Proof. ${}_R\mathfrak{T}$ and ${}_R\mathfrak{R}$ have only the zero module in common by Propositions 1.19, 1.16, 1.10, and 1.15.

Let B be a submodule of A and $A \in {}_R\mathfrak{T}$. The canonical homomorphism from A onto $A/B/T(A/B)$ must have ker A since it would be closed in A. This implies $T(A/B) = A/B$. Hence A/B is a torsion module. Therefore ${}_R\mathfrak{T}$ is closed under taking factors.

${}_R\mathfrak{R}$ is closed under submodules by Proposition 1.2.

Let M be an R-module. Then we have the following exact sequence: $0 \to T(M) \to M \to M/T(M) \to 0$ with $T(M)$ an element of ${}_R\mathfrak{T}$ by Proposition 1.17 and $M/T(M) \in {}_R\mathfrak{R}$ by Proposition 1.16.

Comment. In general, we will use the symbols ${}_R\mathfrak{T}$ and ${}_R\mathfrak{R}$ to represent the full subcategories of ${}_R\mathfrak{M}$ with torsion and torsion free modules, respectively,

2.5 Proposition. If L is a submodule of M, then $T(L) = L \cap T(M)$.

Proof. Using notation as defined in 1.26, we note $L_1 = M_1 \cap L$. Assume $L_\alpha = M_\alpha \cap L$ for all $\alpha < \beta$. If β is a limit ordinal, we have

$$L_\beta = \bigcup_{\alpha<\beta} L_\alpha = \bigcup_{\alpha<\beta} (M_\alpha \cap L) = (\bigcup_{\alpha<\beta} M_\alpha) \cap L = M_\beta \cap L.$$

If β has the predecessor $\beta-1$, then

$$L_\beta = \{m \in M | Rm \subseteq M_{\beta-1}\} \cap L = M_\beta \cap L.$$

Hence $L_\alpha = M_\alpha \cap L$ for all ordinal numbers α. It follows from Proposition 1.27(d) that $T(L) = T(M) \cap L$.

2.6 Proposition. ${}_R\mathfrak{J}$ is closed under taking submodules.

Proof. If $H \in {}_R\mathfrak{J}$ and $A \subset H$, then $T(A) = T(H) \cap A = H \cap A = A$. Therefore A is a torsion module.

2.7 Proposition. ${}_R\mathfrak{R}$ is closed under taking minimal injectives.

Proof. The proof follows from 2.3(3) and 2.6.

2.8 Proposition. If $(\mathfrak{J},\mathfrak{R})$ is a torsion theory on ${}_R\mathfrak{M}$ such that $\{{}_RM | RM = 0\} \subseteq \mathfrak{J}$, then ${}_R\mathfrak{J} \leq \mathfrak{J}$.

Proof. Suppose $M \in {}_R\mathfrak{J} \setminus \mathfrak{J}$, by Proposition 2.1(3). There exist $T \in \mathfrak{J}$ and $F \in \mathfrak{R}$ such that $0 \to T \to M \to F \to 0$ is exact. Since $M \notin \mathfrak{J}$, T is a proper submodule of M by 2.3(6). Let $H = \{m \in M | R_m \subseteq T\}$. T is a proper submodule of H since M contains no proper closed submodules for it was assumed to be an element of ${}_R\mathfrak{J}$. This implies that H/T is a nonzero submodule of F. Hence H/T is an element of $\mathfrak{R}$, since $\mathfrak{R}$ is closed under taking submodules. H/T is an element of $\mathfrak{J}$ since $R(H/T) = 0$. This is a contradiction. Therefore ${}_R\mathfrak{J}$ is contained in $\mathfrak{J}$.

We will now discuss some other torsion theories on ${}_R\mathfrak{M}$ and find conditions on R such that the torsion theories mentioned coincide with $({}_R\mathfrak{T}, {}_R\mathfrak{F})$.

2.9 Let M be any R-module. For $m \in M$, define $(m)_L = \{r \in R | rm = 0\}$.

2.10 After Jans [8], we let $\mathfrak{T}_2 = \{M \in {}_R\mathfrak{M} | \mathrm{Hom}(M, E(R)) = 0\}$ where E(R) denotes the injective hull of R.

In [8], Jans proves:

2.11 Proposition. $\mathfrak{T}_2$ is a torsion class for a torsion theory $(\mathfrak{T}_2, \mathfrak{F}_2)$ on ${}_R\mathfrak{M}$.

2.12 We recall that L, a left ideal of R, is called an <u>essential left ideal of R</u> if L intersects any nonzero left ideal of R nontrivially. We also recall that the <u>singular submodule</u> Z(M) of an R-module M is the set of elements of M which are annihilated by an essential left ideal of R. Z(R) is called the <u>singular ideal</u>.

2.13 After Goldie [5], we let $Z_2(M) = \{m \in M | Em \subseteq Z(M)$ for some essential left ideal E of R$\}$. Then, as noted by Alin and Dickson [1], if we let $\mathfrak{T}_3 = \{M \in {}_R\mathfrak{M} | Z_2(M) = M\}$ and $\mathfrak{F}_3 = \{M \in {}_R\mathfrak{M} | Z_2(M) = 0\}$, then we have:

2.14 Proposition. $(\mathfrak{T}_3, \mathfrak{F}_3)$ is a torsion theory in ${}_R\mathfrak{M}$.

2.15 Proposition. If $(\mathfrak{T}, \mathfrak{F})$ and $(\mathfrak{T}', \mathfrak{F}')$ are torsion theories on ${}_R\mathfrak{M}$, then $\mathfrak{T}$ properly contained in $\mathfrak{T}'$ implies $\mathfrak{F}'$ properly contained in $\mathfrak{F}$, and $\mathfrak{F}$ properly contained in $\mathfrak{F}'$ implies $\mathfrak{T}'$ properly contained in $\mathfrak{T}$.

Proof. The proof follows from Proposition 2.3, (1), (2), (4), and (5).

2.16 Proposition. ${}_R\mathfrak{T}$ is contained in $\mathfrak{T}_3$.

Proof. Let $M \in {}_R\mathfrak{M}$ such that $R(M) = 0$. Then $M = Z(M)$ and hence $M = Z_2(M)$. Hence M is an element of $\mathfrak{T}_3$. The conclusion follows from Proposition 2.8.

2.17 Proposition. If $(\mathfrak{T},\mathfrak{R})$ is a torsion theory on ${}_R\mathfrak{M}$ and $\mathfrak{T}$ is contained in $\mathfrak{T}_2$, then $R \in \mathfrak{R}$.

Proof. If $R \notin \mathfrak{R}$, then there exists T, a left ideal of R, which is contained in $\mathfrak{T}$; but $\mathrm{Hom}(T,E(R)) \neq 0$. Therefore $T \notin \mathfrak{T}_2$.

2.18 Proposition. If $(\mathfrak{T},\mathfrak{R})$ is a torsion theory on ${}_R\mathfrak{M}$ which is closed under taking submodules and $R \in \mathfrak{R}$, then $\mathfrak{T}$ is contained in $\mathfrak{T}_2$.

Proof. If $R \in \mathfrak{R}$ and $(\mathfrak{T},\mathfrak{R})$ is closed under taking submodules, we have $E(R) \in \mathfrak{R}$ by Proposition 2.7. Let $M \in \mathfrak{T}$ and $\alpha \in \mathrm{Hom}(M,E(R))$. Then $\mathrm{Im}\,\alpha \in \mathfrak{R}$ since $E(R) \in \mathfrak{R}$ and $\mathfrak{R}$ is closed under taking submodules. Hence $\mathrm{Im}\,\alpha = 0$ and so $\alpha = 0$. This implies that $M \in \mathfrak{T}_2$. Since M was an arbitrary element of $\mathfrak{T}$, we have that $\mathfrak{T}$ is contained in $\mathfrak{T}_2$.

2.19 Proposition. ${}_R\mathfrak{T}$ is contained in $\mathfrak{T}_2$ if and only if $T(R) = 0$.

Proof. The proof follows from Propositions 2.17 and 2.18.

Note that if $Z_2(R) \neq 0$, then $Z_2(R) \in \mathfrak{T}_3 \setminus \mathfrak{T}_2$ since $\mathrm{Hom}(Z_2(R),E(R)) = 0$ if and only if $Z_2(R) = 0$. In fact, we have the following:

2.20 Proposition. $\mathfrak{T}_3 = \mathfrak{T}_2$ if and only if $Z(R) = 0$.

Proof. See Gentile [4].

Examples.

(1) If R is the ring of integers, then $Z(R) = 0$; hence $\mathfrak{J}_3 = \mathfrak{J}_2$. Note $R(R/(n)) \in \mathfrak{J}_3$ for $R(R/(n)) \in {}_R\mathfrak{R}$ for any $n \in R$.

(2) If R is the ring of integers modulo 12, then $Z(R) \neq 0$. However, $T(R) = 0$. Hence $0 \neq Z_2(R) \in \mathfrak{J}_3 \cap {}_R\mathfrak{R} \cap \mathfrak{R}_2$.

2.21 Definition. For L a submodule of M, we define the closure of L in M, cl L_M, to be the intersection of all closed submodules of M containing L. (Recall that $N \subset M$ is closed if $M/N \in {}_R\mathfrak{R}$.)

2.22 Proposition. If L is a submodule of M, then cl L_M is a closed submodule of M.

2.23 Definition. We call a ring R full if the closure in R of each essential left ideal of R is R. That is, R is full if and only if it does not contain a proper essential closed left ideal.

2.24 Proposition. R is a full ring if and only if for each nonzero element m of a torsion free module M, $(m)_L$ is not an essential left ideal of R.

Proof. Assume that R is a full ring, Note that $(m)_L$ is a closed proper submodule of R if m is a nonzero element of a torsion free module M since $(m)_L = \ker \alpha_m$ where $\alpha_m \in \mathrm{Hom}(R,M)$ defined by $\alpha_m(r) = rm$. Therefore $(m)_L$ is not an essential submodule of R, since it is not equal to R and it is closed.

Conversely we assume for each nonzero element m of a torsion free module M, $(m)_L$ is not an essential left

ideal of R. Suppose R is not a full ring, then R contains a proper closed essential left ideal L. Let $r' \in R \backslash L$ and $H = \{r \in R \mid rr' \in L\}$. Then H is easily seen to be an essential left ideal of R. However, $H \subseteq (r + L)_L$ where $(r + L)$ is an element of the torsion free module R/L. This contradicts the assumption. Hence we see that R is a full ring.

2.25 Proposition. R is a full ring if and only if $\mathfrak{R}_3 = {}_R\mathfrak{R}$.

Proof. Assume R is a full ring and let $F \in {}_R\mathfrak{R}$. Note that for $0 \neq m \in F$, $(m)_L$ is a proper closed left ideal of R. From this we see that $(m)_L$ is not an essential left ideal of R. Therefore, $Z_2(F) = 0$, which implies $F \in \mathfrak{R}_3$. It follows, from Propositions 2.3 (6) and 2.15, that ${}_R\mathfrak{R} = \mathfrak{R}_3$.

Conversely, we assume $\mathfrak{R}_3 = {}_R\mathfrak{R}$ and let L be an essential closed left ideal of R. Then each nonzero element of R/L is annihilated by an essential left ideal of R (as noted in the proof of Proposition 2.24). It follows that R/L is an element of $\mathfrak{T}_3$ and ${}_R\mathfrak{R}$. Hence $R/L = 0$ since ${}_R\mathfrak{R} = \mathfrak{R}_3$. Therefore $L = R$ which implies that R is a full ring.

2.26 If $T(R) = 0$ and R is a full ring, then ${}_R\mathfrak{T} = \mathfrak{T}_2$.

Proof. By Proposition 2.19, ${}_R\mathfrak{T}$ is contained in $\mathfrak{T}_2$ if $T(R) = 0$.

Assume $T(R) = 0$ and R is a full ring.

Suppose $M \notin {}_R\mathfrak{T}$, then $M/T(M)$ is a nonzero torsion free module. Let $m + T(M)$ be a nonzero element of $M/T(M)$.

Then there exists $r \in R$ such that $r(m + T(M)) \neq 0$ and $Rr \cap (m + T(M))_L = 0$. Hence the homomorphism α from Rr onto $Rr(m + T(M))$ defined by $\alpha(r'r) = r'r(m + T(M))$ for $r'r \in Rr$ is an isomorphism. Consider the following diagram of homomorphisms.

$$\begin{tikzcd} 0 \arrow[r] & Rr(m + T(M)) \arrow[r, "j"] \arrow[d, "\alpha^{-1}"'] & M/T(M) \arrow[dl, dashed, "\sigma"] \\ & E(R) & \end{tikzcd}$$

where j is the injection. There exists σ from $M/T(M)$ into $E(R)$ such that $\sigma_j = \alpha^{-1}$ since $E(R)$ is an injective module. σ is a nonzero homomorphism since α^{-1} is nonzero. Therefore M is not an element of $\mathcal{F}_2$. Hence $\mathcal{F}_2$ is contained in ${}_R\mathcal{F}$. Therefore $\mathcal{F}_2 = {}_R\mathcal{F}$.

2.27 <u>Proposition</u>. If $Z(R) = 0$, then R is a full ring if and only if ${}_R\mathcal{F} = \mathcal{F}_2$.

<u>Proof</u>. This follows from Propositions 2.20 and 2.25.

We will now consider the function T as an operator on the category of rings. Note that while T is a subfunctor of the identity functor on ${}_R\mathfrak{M}$, it is not on the category of rings. Let, e.g., S be a nilpotent subring of a ring R with an identity, then $T_S(S) = S$, but $T_R(R) = 0$. Hence, to emphasize this distinction, when we think of T as an operator on the category of rings, we write

$$H(R) = T_R({}_RR).$$

Recall that $H(R)$ is a two-sided ideal of R.

Obviously, we have

2.28 Proposition. $H(R) = 0$ if and only if $r(R) = 0$.

2.29 Proposition. If R and S are rings and f is a ring homomorphism from R onto S, then $f(H(R)) \subseteq H(S)$.

Proof. Consider $R \xrightarrow{f} S \xrightarrow{g} S/H(S)$ where g is the natural ring homomorphism. Note $S/H(S)$ is a torsion free S-module. It is easily seen that $S/H(S)$ is a torsion free R-module where $r(S + H(S)) = f(r)(S + H(S))$. This implies that the ker fg contains $H(R)$. (Note f and g are now being considered R-module homomorphisms and S, as an R-module.) Hence $f(H(R))$ is contained in $H(S)$.

Comment. If f is not onto, we can conclude that $f(H(R)) \subseteq H(f(R))$. However, as we see from the example mentioned above, we cannot conclude that $f(H(R)) \subseteq H(S)$.

2.30 Corollary. If $H(R) = R$ and S is a (ring) homomorphic image of R, then $H(S) = S$.

2.31 Definition. R will be said to be H-torsion if $H(R) = R$. The class of H-torsion rings will be designated by $\mathfrak{H}$.

2.32 Proposition. $\mathfrak{H}$ is closed under taking extensions.

Proof. Consider $0 \longrightarrow J \longrightarrow S \longrightarrow K \longrightarrow 0$, an exact sequence of rings with $H(J) = J \neq 0$ and $H(K) = K \cong S/J \neq 0$. If either end is zero, the proof is immediate. We want to show that $H(S) = S$. If $(s + J) \in (S/J)_1$, then $sr \in S_1$ for all $r \in J_1$ since $s's$ is an element of J for all $s' \in S$. Suppose $(s + J) \in (S/J)_\alpha$

implies $sr \in S_\alpha$ for all $r \in J_1$ and all $\alpha < \beta$. If β is a limit ordinal and $(s + J) \in (S/J)_\beta$, then $(s + J) \in (S/J)_\alpha$ for some $\alpha < \beta$, which implies $sr \in S_\alpha \leq S_\beta$ for all $r \in J_1$. If β has a predecessor $\beta - 1$, then if $s + J \in (S/J)_\beta$, then $S/J(s + J) \subseteq (S/J)_{\beta-1}$, which implies $s's + J \in (S/J)_{\beta-1}$ for all $s' \in S$. Hence $s'sr \in S_{\beta-1}$ for all $r \in J_1$ and all $s' \in S$ by the inductive assumption. Therefore, $sr \in S_\beta$ for all $r \in J_1$. It follows from the above induction that SJ_1 is contained in $T_s(S)$ since $H(S/J) = S/J$ and $JJ_1 = 0$.

If $SJ_1 = 0$, then J_1, which is not equal to zero, is contained in S_1. If $SJ_2 \neq 0$, then $T_s(S) \neq 0$, which implies S_1 is not equal to zero. Hence S_1 is not equal to zero.

Suppose $S_\beta = S_{\beta+1}$ and S_β is properly contained in S for some ordinal β. By 1.27, $S_\beta = H(S)$, which is an ideal of S by 1.23. Getting $(J + S_\beta)/S_\beta$ equal to J', we have $H(J') = J'$ by Proposition 2.29 since J' is isomorphic to $J/J \cap S_\beta$ and $H(J) = J$. Writing S' for S/S_β, we note that J' is an ideal of S' and S'/J' is isomorphic to $S/J+S_\beta$. Hence $H(S'/J') = S'/J'$ since S'/J' is a homomorphic image of S/J. Hence we have $0 \longrightarrow J' \longrightarrow S' \longrightarrow S'/J' \longrightarrow 0$ is an exact sequence of rings with the ends H-torsion. If either end is zero, then $H(S') = S' \neq 0$ which implies that $S_1' \neq 0$. If the ends are not zero, then it follows from the above induction that $S_1' \neq 0$. Noting that

$$S_1' = \{s + S_\beta | S/S_\beta(0 + S_\beta) = 0\} \neq 0$$

it follows that $S_\beta \neq S_{\beta+1}$. This contradiction implies that

$T_S(S) = H(S) = S$.

2.33 Proposition. $\mathcal{H}$ is closed under taking direct sums.

Proof. Let $\{R_a\}_{a \in A}$ be a collection of rings such that $H(R_a) = R_a$ for all $a \in A$ and consider $R = \oplus \sum_{a \in A} R_a$. It is easily seen by transfinite induction that $(R'_a)_\alpha$ is contained in R_α for any ordinal α and for all $a \in A$ where $(R'_a)_\alpha = \pi_a(R_a)_\alpha$, π_a the injection from R_α into R. Hence R'_a is contained in H(R) for all $a \in A$. Therefore H(R) = R.

Example. If $R = \prod_{n=2}^{\infty} (2)/(2^n)$, then $H(R) \neq R$ since R is not a nil ring, but $H(2)/(2^n) = (2)/(2^n)$ for all n. Therefore direct products of H-torsion rings are not necessarily H-torsion.

2.34 Proposition. $\mathcal{H}$ is closed under taking subobjects.

Proof. Let K be a subring R and H(R) = R. From Proposition 2.5 we note that $T_R(K) = K \cap T_R({}_RR) = K$. It is easily shown that K_α when considered an R-module is contained in K_α when considered a K-module for all ordinal numbers α. Hence $T_R(K)$ is contained in $T_K(K)$. Therefore H(K) = K.

We will now show that $\mathcal{H}$ is not a class of radical rings for any radical property. First we recall some definitions. (See [3, Chapter 1].)

Let S be a property a ring may possess. A ring R is called an S-ring if it possesses the property S. An ideal J of R is an S-ideal if J is an S-ring. A ring which does not contain any nonzero S-ideals is called S-semi-simple.

S is called a radical property if the following conditions hold:

(1) A homomorphic image of an S-ring is an S-ring.

(2) Every ring contains an S-ideal K which contains every other S-ideal of the ring.

(3) The factor ring R/K is S-semi-simple.

For a given radical property S, the class of S-rings is called the S-radical class of rings.

Let $\mathfrak{D}$ be any nonempty class of rings. A ring is said to be of first degree over $\mathfrak{D}$ if it is a homomorphic image of some ring in $\mathfrak{D}$.

Assume that rings of degree α over $\mathfrak{D}$ have been defined for every $\alpha < \beta$. A ring R is said to be of degree β over $\mathfrak{D}$ if every nonzero homomorphic image of R contains a nonzero ideal which is of degree α over $\mathfrak{D}$, for some $\alpha < \beta$.

Let $\bar{\mathfrak{D}}$ be the class of all rings which are of any degree over $\mathfrak{D}$. Define a ring to be an $S_{\mathfrak{D}}$ ring if R is an element of $\bar{\mathfrak{D}}$.

2.35 Proposition. $S_{\mathfrak{D}}$ is a radical property and $\mathfrak{D}$ is contained in $\bar{\mathfrak{D}}$. ([3, p. 12.]) $S_{\bar{\mathfrak{D}}}$ is called the lower radical property determined by $\mathfrak{D}$.

2.36 Proposition. If $\mathfrak{J}$ is a radical property and every ring in $\mathfrak{D}$ is a $\mathfrak{J}$-radical ring, then the lower radical property determined by $\mathfrak{D}$ is contained in $\mathfrak{J}$, i.e., if $R \in \bar{\mathfrak{D}}$ then R is a $\mathfrak{J}$-radical ring. ([3, p. 13.])

2.37 We recall that if $\mathfrak{D}$ is the class of nilpotent rings then $S_{\mathfrak{D}}$ is called the Baer Lower radical property

(frequently called the Baer-McCoy radical), (see e.g., [3], p. 59).

2.38 Proposition. If $H(R) = R$, then R is of second degree over $\mathfrak{Z}$ where $\mathfrak{Z}$ is the class of all zero rings. Hence all H-torsion rings are Baer-McCoy radical rings.

Proof. Let $H(R) = R$ and let S be a homomorphic image of R. Then $H(S) = S$ by 2.30. Then S_1 is the desired nilpotent ideal of S.

2.39 Proposition. The class of Baer-McCoy radical rings is the smallest class of radical rings which contains $\mathfrak{N}$.

Proof. This proposition follows from 2.36 and 2.38 when one notes that $\mathfrak{N}$ contains all nilpotent rings.

Remarks. An example of a Baer-McCoy radical ring which is not an H-torsion ring is given by any Baer-McCoy radical ring of degree greater than two over the nilpotent rings.

However, there are even Baer-McCoy radical rings of degree 2 over the nilpotent rings which are H-torsion free. For example, the commutative algebra A over the field of real numbers with basis $x_\alpha \, x_\beta = x_{\alpha+\beta}$ if $\alpha + \beta < 1$ and $x_\alpha \, x_\beta = 0$ if $\alpha + \beta \geq 1$ (see [3, p. 19]).

Since the ring A is commutative and nil, each element generates a nilpotent ideal. It follows that every homomorphic image of A also has this property; so A is of degree 2 over the class of nilpotent rings. Further, the right annihilator of A is 0 and therefore $H(A) = 0$.

It is also of interest to note that every nonzero (left) ideal of A is essential and that the left annihilator of every element is not zero. Hence $A = Z(S)$ and A is a Goldie-torsion ring.

Since the Baer-McCoy radical of a ring R is the intersection of those ideals Q of R such that R/Q has no nilpotent ideals, ([3, p. 56]), it is clear that the Baer-McCoy radical is closed in R.

SECTION III

THE EXTENSION OF THE STRUCTURE THEOREM FOR SEMI-SIMPLE ARTINIAN RINGS TO THE CATEGORY ${}_R\mathfrak{R}$

The principal objective of this section is to prove that the following conditions are equivalent for Jacobson semi-simple rings:

(a) All torsion free R-modules are projective in ${}_R\mathfrak{R}$;

(b) All torsion free R-modules are injective in ${}_R\mathfrak{R}$;

(c) R has d.c.c. on closed left ideals;

(d) R is semi-simple Artinian.

In general, we show that if R is any ring such that $T(R) = 0$ and all torsion free R-modules are projective, then ${}_RR = {}_RJ \oplus {}_RS$, where J is the Jacobson radical of R and S is a semi-simple Artinian subring of R. It is an open question whether or not $J = 0$ under these conditions. If in the above theorem R is assumed nil semi-simple, then $R = J \oplus S$ is a ring direct sum. The above theorems are shown to apply to the torsion theories of A. W. Goldie and J. P. Jans under suitable conditions on R.

3.1 Recall that a ring R is said to be a <u>dense ring of</u> linear transformations on V such that for any finite linearly independent set $x_1, x_2, \ldots, x_n$ in V and any finite arbitrary

set $y_1, y_2, \ldots, y_n$ in V, there exists an element $r \in R$ such that $r(x_i) = y_i$ for $i = 1, 2, 3, \ldots, n$. Such a ring is called primitive.

For the purposes of this paper, when we speak of R as a primitive ring it will be understood that R is a dense subring of the full ring of linear transformations on the vector space V over the division ring D. The rank of $r \in R$ is understood to mean the dimension of $r(V)$.

3.2 Proposition. If R is a primitive ring, $r \in R$, and rank $r < \infty$, then there exists $r' \in R$ such that $r'r = r$.

3.3 Proposition. If R is a primitive ring and $r \in R$, then rank $r'r \leq$ rank r.

3.4 Proposition. If R is a primitive ring, then L is a simple left ideal of R if and only if $L = Rg$ where rank $g = 1$.

3.5 Proposition. If L is a simple left ideal of the primitive ring R, then L is a closed left ideal of R.

Proof. Assume L is a simple left ideal of R; then, by Proposition 3.4, $L = Rg$ where rank of $g = 1$. Note that if r is a nonzero element of L, then rank $r = 1$ by Proposition 3.3. Let $0 \neq h \in R$ such that Rh is contained in L. It is easily seen that rank $h = 1$. Hence, by Proposition 3.2, $h \in Rh$ which is contained in L. Therefore, L is a closed left ideal of R by Proposition 1.11.

3.6 Recall that the Socle (Soc M) of ${}_RM$ is defined to be the sum of all simple submodules of M.

3.7 Proposition. If Soc R $\neq 0$, R is a primitive ring, and U is an infinite dimensional subspace of V, then there exists $\{e_j\}_{j\in I} \subseteq R$ such that $e_j e_i = \delta_{ij} e_i$ for all e, $j \in I$, I is infinite, and $e_i(V) = [w_i]$ where $w_i \in U$ for all $e \in I$.

Proof. Assume Soc R $\neq 0$ and U is an infinite dimensional subspace of V. By 3.4, there exists $g \in R$ such that rank $g = 1$ since R contains simple left ideals. Let v_1 and v_2 be nonzero vectors such that $g(v_1) = v_2$ and w_1 a nonzero vector in U. Since R is a dense ring of linear transformations, there exists f, $h \in R$ such that $f(w_1) = v_1$ and $h(v_2) = w_1$. Since the rank $g = 1$, we have $hgf(V) = [w_1]$ and $(hgf)^2 = hgf$. Ker $hgf \cap U$ is an infinite dimensional subspace since $U = \ker hgf \cap U \oplus hgf(U)$ and $hgf(U)$ is a one-dimensional space. We have shown that there exists $\{e_1\} \subseteq R$ such that $e_1 e_1 = e_1$, $\ker e_1 \cap U$ is infinite dimensional and $e_1(V) = [w_1]$ where $w_1 \in U$.

Assume there exists $\{e_1, \ldots, e_n\} \subseteq R$ such that $e_i e_j = \delta_{ij} e_i$, i, $j \in \{1, \ldots, n\}$. $\bigcap_{j=1}^{n} \ker e_j \cap U$ is an infinite dimensional space, $e_i(V) = [w_i]$, $0 \neq w_i \in U$, $i = 1, \ldots, n$. It follows from the orthogonality of idempotent elements $\{e_1, \ldots, e_n\}$ that $\{w_1, \ldots, w_n\}$ is a linearly independent set of vectors. Let $0 \neq w_{n+1} \in M = \bigcap_{j=1}^{n} \ker e_j \cap U$. Now $\{w_1, w_2, \ldots, w_n, w_{n+1}\}$ is a linear independent set since $M \cap [w_1, \ldots, w_n] = 0$. As in the first paragraph of the proof, it can be shown that there exists e_{n+1} an idempotent element of rank 1 in R such that

$e_{n+1}(w_j) = \delta_{j,j}\, w_j$, $j = 1, \ldots, n + 1$. This implies $e_j\, e_{n+1} = e_{n+1}\, e_j = \delta_{n+1,j}\, e_j$, $j = 1, \ldots, n + 1$. Ker $e_{n+1} \cap M$ is an infinite dimensional subspace since $M = \ker e_{n+1} \cap M \oplus e_{n+1}(M)$ and $e_{n+1}(M)$ is a one-dimensional space. Hence, by induction, we can form a chain $\{e_1\}_1 \subseteq \{e_1, e_2\}_2 \subseteq \{e_1, e_2, e_3\}_3 \subseteq \ldots$ where $\{e_1, \ldots, e_n\}_n$ satisfies the conclusion of the proposition with the exception that the set is finite. Let $B = \bigcup_{m=1}^{\infty} \{e_1, \ldots, e_m\}_m$. Clearly, B satisfies the conclusion of the proposition.

3.8 Proposition. Let R be any ring such that $T(R) = 0$. Let $\{e_i\}$ $i \in I$ be an infinite family of pairwise orthogonal idempotents in R. If the sets $P_r = \{e_i \mid 0 \neq e_i r \in Ze_i\}$ and ${}_rP = \{e_i \mid 0 \neq re_i \in Ze_i\}$ are finite for all $r \in R$, then ${}_R\mathfrak{R}$ contains a module which is not $\mathfrak{R}$-projective.

Proof. Let $\{e_j\}_{j \in I}$ be an infinite set of orthogonal idempotent elements in R satisfying the hypothesis of the proposition. Let I^n be the set of j in I such that $nej = 0$ where n is a fixed element of Z. Partition I into I_1 and I_2 such that I_1 and I_2 are infinite and, if I^n is infinite, then $I^n \cap I_1$ and $I^n \cap I_2$ are infinite.

Define $\alpha : R^1/T(R^1) \longrightarrow \prod_{j \in I} Re_j$ by $\alpha(\overline{0,1}) = h, ((\overline{0,1}) = (0,1) + T(R^1))$ where $h(j) = e_j$ if $j \in I_1$, and $h(j) = 0$ if $j \in I_2$, and extend α to a homomorphism.

We shall show that $M = \operatorname{Im} \alpha$ is not $\mathfrak{R}$-projective. If it were, there would exist $\theta : M \longrightarrow R^1/T(R^1)$ such that $\alpha\, \theta = 1$. Let $\theta(h) = (\overline{r,n})$. Then $h = \alpha\, \theta(h) = \alpha(\overline{r,n}) = \alpha(r(0,1) + n(0,1)) = rh + nh$. Thus $rh = (1-n)h$, and this

implies that $re_j = (1-n)e_j$ for $j \in I_1$. It follows from the hypothesis of the proposition that $(1-n)e_j = 0$ for all but a finite number of the j's in I_1. Hence $(1-n)e_j = 0$ for an infinite number of j's in I_1. By construction of I_1 and I_2, it follows that $I^{n-1} \cap I_1$ and $I^{n-1} \cap I_2$ are infinite sets. Note $e_j h = 0$ if $j \in I_2$. For $j \in I^{n-1} \cap I_2$, we have the following equalities:

$0 = \theta(e_j h) = e_j \theta(h) = e_j \overline{(r,n)} = \overline{(e_m r + ne_j, 0)}$. Hence, by Corollary 1.37, $R(e_j r + ne_j) = 0$. Hence, for $j \in I^{n-1} \cap I_2$, $0 = e_j(e_j r + ne_j) = e_j r + ne_j = e_j r + e_j$ since $(n-1)e_j = 0$ for $j \in I^{n-1} \cap I_2$. This implies $e_j r = -e_j$ for $j \in I^{n-1} \cap I_2$, which is impossible since $I^{n-1} \cap I_2$ is an infinite set, and this is contrary to the hypothesis of the proposition. Therefore M is non $\mathfrak{R}$-projective.

3.9 Proposition. The following are equivalent:

(1) All torsion free modules are $\mathfrak{R}$-projective.

(2) If L is a closed submodule of a torsion free module M, then L is a direct summand of M.

(3) If M, N $\in {}_R\mathfrak{R}$ and α is a homomorphism from M onto N, then M = ker $\alpha \oplus$ N' where N' is isomorphic to N.

3.10 Proposition. If all R-modules in ${}_R\mathfrak{R}$ are $\mathfrak{R}$-projective and R is a primitive ring, then Soc R $\neq 0$.

Proof. Assume that all R-modules in ${}_R\mathfrak{R}$ are $\mathfrak{R}$-projective and that R is a primitive ring. Note that for nonzero v in V, V = Rv is a simple torsion free R-module. Let v be any nonzero element of V. Define $\alpha: R \to Rv$ by $\alpha(r) = rv$ for $r \in R$. Ker α is a closed submodule of R

since Rv is torsion free. Hence, by Proposition 3.9, $R = \ker \alpha \oplus L$ where L is a simple R-module since L is isomorphic to Rv. Therefore R contains a simple left ideal which implies that $\mathrm{Soc}\, R \neq 0$.

3.11 <u>Proposition</u>. The Socle of R is the set of linear transformations of finite rank contained in R.

<u>Proof</u>. (See [7], p 75.)

3.12 <u>Proposition</u>. If all torsion free modules are projective and R is a primitive ring, then $\mathrm{Soc}\, R$ is a closed left ideal of R.

<u>Proof</u>. Assume R is a primitive ring and all torsion free modules are $\mathcal{R}$-projective. Suppose that $\mathrm{Soc}\, R$ is not closed in R. Then there exists $h \in R \backslash \mathrm{Soc}\, R$ such that $rh \in \mathrm{Soc}\, R$ for all $r \in R$. It follows from Proposition 3.11 that rank $h = \infty$ and rank $rh < \infty$ for all $r \in R$. Let $h(V) = U$. Then, from Propositions 3.10 and 3.7, there exists an infinite set of orthogonal idempotent elements $\{e_j\}_{j\in I}$ contained in R where the image space of e_j is a one-dimensional subspace of U. Since $\{e_j\}_{j\in I}$ is an infinite set of orthogonal idempotent elements of rank 1, there exists a set of linearly independent vectors H whose elements can be indexed by I such that for $w_j \in H$, $e_j(w_j) = w_j$.

Let $r \in R$ and define ${}_rP = \{e_j | re_j = c_j e_j \neq 0,\ c_j \in Z\}$. Note if $e_j \in {}_rP$, then $r(w_j) = r(e_j(w_j)) = re_j(w_j) = c_j e_j(w_j) = c_j w_j \neq 0$. Hence $\{c_j w_j | e_j \in {}_rP\}$ is a subset of $rh(V)$ and therefore must be finite since $rh(V)$ is finite dimensional. Therefore ${}_rP$ is finite for all r in R.

Let $P_r = \{e_j | e_j r = c_j e_j \neq 0$ for some c_j in $Z\}$. Note that for $e_j \in P_r$ we have $e_j r(w_j) = c_j e_j(w_j) = c_j(w_j) \neq 0$. It follows that $\{r(w_j) | e_j \in P_r\}$ is linearly independent. Suppose $\sum \alpha_j e_k r(w_j) = \sum \alpha_j c_k e_k(w_j) = \alpha_k c_k w_k = 0$. Since $c_k w_k \neq 0$, we have $\alpha_k = 0$.

Now, since w_j is in $h(V)$, $r(w_j)$ is in $rh(V)$, which, as noted above, is finite dimensional. It follows that P_r is finite.

Thus ${}_rP$ and P_r are finite for all r in R and so, by 3.9, there exists $M \in {}_R\mathfrak{R}$ which is not $\mathfrak{R}$-projective. This contradicts the assumption. Hence Soc R is a closed left ideal of R.

3.13 Proposition. If all torsion free modules are projective and R is a primitive ring, then R is simple Artinian.

Proof. Assume all torsion free modules are projective and R is a primitive ring. By Propositions 3.10 and 3.12, Soc R is a nonempty closed left ideal of R. From Proposition 3.9, ${}_RR = {}_R\mathrm{Soc}\ R \oplus K$. Since Soc R is an ideal of R, and Soc R $\neq 0$, this implies that $K = 0$. Hence $R = \mathrm{Soc}\ R$.

Suppose V is infinite dimensional. Then R contains an infinite set of orthogonal idempotent elements $\{e_j\}_{j\in I}$ of rank 1 by Proposition 3.7. For $r \in R$, $r(V)$ is finite dimensional since Soc R = R. We arrive at a contradiction by essentially repeating the second half of the proof of Proposition 3.12. Therefore V is finite dimensional.

The conclusion follows since R was assumed to be a dense ring of linear transformations.

3.14 Recall that a ring R is said to be a <u>subdirect sum</u> of a family of rings $\{R_i | i \in I\}$ if R is a subring of $\prod_{i \in I} R_i$ and, for each $i \in I$, the projection π_i restricted to R is onto R_i.

3.15 <u>Proposition</u>. If all $M \in {}_R\mathfrak{R}$ are $\mathfrak{R}$-projective and R is a subdirect sum of a family $\{R_i | i \in K\}$ of simple Artinian rings, then R is a semi-simple Artinian ring.

<u>Proof</u>. Assume that all $M \in {}_R\mathfrak{R}$ are $\mathfrak{R}$-projective and R is a subdirect sum of the family $\{R_i | i \in K\}$ of simple Artinian rings. Let π_i be the projection homomorphism from R onto R_i for each $i \in K$. Note π_i can be considered a ring homomorphism or an R-module homomorphism with R and $R_i \in {}_R\mathfrak{R}$. By the assumption that all torsion free modules are projective, we have ${}_RR = {}_R(\ker \pi_j) \oplus {}_RR^j$ where ${}_RR^j \cong {}_RR_j$, for each $j \in K$.

We now show that ${}_RR = {}_R(\ker \pi_j) + {}_RR^j$ is, i.e., $\ker \pi_j$ and R^j are ideals of R. Ker π_j is an ideal of R since π_j can be considered a ring homomorphism. $(\text{Ker } \pi_j)R^j = 0$ since $\ker \pi_j$ is an ideal, R^j is a left ideal, and $\ker \pi_j \cap R^j = 0$. This implies $(R^j(\ker \pi_j))^2 = 0$. Note that $R^j(\ker \pi_j)$ is an ideal. Hence $R^2(\ker \pi_j) = 0$ since R is Jacobson semi-simple and, hence, also semi-simple. It follows that R^j is an ideal of R. Therefore $R = (\ker \pi_j) + R^j$ as a ring direct sum.

We note that R^j is ring isomorphic to R_j since π_j

restricted to R^j is an isomorphism onto R_j. Hence R^j is also a simple Artinian ring. Let e_j denote the identity of R^j. Since R^j is an ideal of R, we have $e_jRe_j = e_jR = Re_j = R^j$. Further, each e_j is in the center of R. Since each R^j is simple, if $R^i \neq R^j$, then $R^iR^j = 0$. It follows that $e_ie_j = 0$. Let $\aleph$ denote the set $\{e_j\}$ of all such idempotents. One verifies immediately that $J = \sum e_jRe_j = \sum R^j$ is a direct sum. Furthermore, J is an ideal since each R^j is an ideal. Note that it is possible that, for $i \neq j$, we might have $R^i = R^j$; but, nevertheless, there is a one-to-one correspondence between the set of ideals R^j and $\aleph$.

We now observe that an element r in R is contained in J if and only if the set $\{e_j \in \aleph | e_jr \neq 0\}$ is finite. If $r \in J$, $r = e_{j_1}r_{j_1} + \ldots + e_{j_n}r_{j_n}$, and so it is clear that $e_jr = 0$ except when j is one of $j_1, \ldots, j_n$.

Conversely, assume that $e_jr \neq 0$ if and only if $e_j \in \{e_{j_1}, \ldots, e_{j_m}\}$. Suppose $0 \neq s = e_{j_1}r + \ldots + e_{j_m}r - r$. Since $s = 0$, there exists π_k such that $\pi_k(s) \neq 0$. Hence there exists $r_k \in R^k$ such that $\pi_k(r_k) = \pi_k(s)$. Note that $0 \neq \pi_k(r_k) = \pi_k(e_kr_k) = \pi_k(e_k)\pi(r_k) = \pi(e_k)\pi_k(s) = \pi_k(e_ks)$ $= \pi_k(e_ke_{j_1}r + \ldots + e_ke_{j_m}r - e_kr) = \pi_k(0) = 0$. This contradiction proves that if $e_jr \neq 0$ for only finitely many e_j's in $\aleph$, then $r \in J$.

It follows from the above observation that, if $\aleph$ is a finite set, then R is semi-simple Artinian.

We now assume that $\aleph$ is an infinite set. We wish to show that J is closed in R. If $J = R$, then it is closed in

R. Assume $J \neq R$ and let r be an arbitrary element in R which is not in J. It is enough to prove that Rr is not contained in J. Note that $I = \{e_j | e_j r \neq 0;\ e_j \in \aleph\}$ is not a finite set since $r \notin J$. For p, a positive prime number, let $K_p = \{e_j \in I | pe_j = 0\}$.

Let S denote the set of primes p such that K_p is finite and non-empty. Note that since e_j is the identity of a semi-simple Artinian ring, its characteristic is prime. Hence $I = \cup K_p$ is a disjoint union. We now partition I into I_1 and I_2 so that

(i) If K_p is finite, then $K_p \subset I_1$ or $K_p \subset I_2$.

(ii) If S is infinite, both I_1 and I_2 contain infinitely many K_p.

(iii) If K_p is infinite, then $I_1 \cap K_p$ and $I_2 \cap K_p$ are both infinite.

Now, as in the proof of 3.8, we define $\alpha: R^1/T(R^1) \longrightarrow \pi Re_j$ by $\alpha(\overline{0,1}) = h$ where $h(j) = e_j$ for $e_j \in I_1$ and $h(j) = 0$ for $j \in I_2$. By hypothesis, $M = \operatorname{Im} \alpha$ is projective and so, as in 3.8, we obtain $0 \neq s$ in R and k in Z such that

(1) $se_j = (1-k)e_j,\ e_j \in I_1$

(2) $e_j s = -ke_j,\ e_j \in I_2$

We now claim that, for infinitely many e_j in I, $se_j = k_j e_j \neq 0$. First, if $k = 0$, this is clear from (1); so we may assume $k \neq 0$. There are two cases. First, suppose S is infinite. In this case, by (i) and (ii), there are infinitely many primes p such that $K_p \subset I_2$ and $p > k$.

If $e_j \in K_p$ for such p, since $k \neq 0$ we have $e_j s = -ke_j \neq 0$. Next, suppose S is finite. Then there is a q such that K_q is infinite and, by (iii), $K_q \cap I_1$ and $K_q \cap I_2$ are both infinite. If $q|k$, $e_j = se_j$ for $e_j \in I_1 \cap K_q$ by (1). If $(q,k) = 1$, $e_j s = -ke_j \neq 0$ for $e_j \in I_2 \cap K_q$ by (2).

Now it follows from the above paragraph and the fact that each e_j is in the center of R that $e_j sr = se_j r = k_j e_j r \neq 0$ for infinitely many e_j in I. Hence sr is not in J and so J is closed.

It follows that ${}_R R = {}_R J \oplus {}_R H$ since by assumption all torsion free modules are $\mathfrak{R}$-projective. Suppose $H \neq 0$ and $0 \neq r \in H$, then there exists $i \in K$ such that $\pi_i(r) \neq 0$, from which it follows that $e_i r \neq 0$. However, $e_i r \in J \cap H$ since J is an ideal and H is a left ideal. This is a contradiction. Hence $H = 0$ and $R = J$.

It follows from the above characterization of J that $\mathfrak{H}$ is a set of idempotent elements of R satisfying the hypotheses of Proposition 3.7. This implies that there exists a torsion free R-module M which is not $\mathfrak{R}$-projective. It follows from this contradiction that $\mathfrak{H}$ is finite. Therefore R is Artinian semi-simple.

3.16 <u>Proposition</u>. If all torsion free R-modules are projective and S is a homomorphic image of R, then all torsion free S modules are $\mathfrak{R}$-projective.

<u>Proof</u>. Any S-module M can be considered a torsion free R-module by defining $rm = f(r)m$ where $r \in R$, $m \in M$, and f is the homomorphism from R onto S. It is easily

seen that, for M and N torsion free S-modules, $\alpha: M \to N$ is an S-module homomorphism and α is an R-module homomorphism. The proposition follows immediately from this observation.

3.17 Proposition. If R is a Jacobson semi-simple ring and all torsion free modules are $\Re$-projective, then R is Artinian semi-simple.

Proof. Assume R is a Jacobson semi-simple ring and all torsion free modules are $\Re$-projective. From [page 102 of 3], R is a subdirect sum of a family $\{R_i \mid i \in I\}$ of dense rings of linear transformations on vector spaces V_i over division rings D_i. Since the projections π_j are onto R_j for $j \in I$, we have that all torsion free R_j-modules are $R_j\Re$-projective for $j \in I$. By 3.13, R_j is a simple Artinian ring for each j in I. From Proposition 3.15, R is a semi-simple Artinian ring.

3.18 Proposition. The following are equivalent if R is a Jacobson semi-simple ring:

(1) $M \in {}_R\Re$ implies M is $\Re$-projective.

(2) $M \in {}_R\Re$ implies M is $\Re$-injective.

(3) The closed submodules of R have d.c.c.

(4) R is semi-simple Artinian.

Proof. Clearly (4) implies (1), (2), (3), (4).

(1) implies (4) is Proposition 3.16.

Assume (2) and R is a Jacobson semi-simple ring. Note that $T(R) = 0$ by Proposition 1.34. Consider the following diagram of homomorphisms:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "j"] \arrow[d, "i"'] & R^1/T(R^1) \arrow[dl, dashed, "\alpha"] \\ & R & \end{tikzcd}$$

$j(r) = (r,0) + T(R^1)$ which is a monomorphism by 1.38, $i(r) = r$, and α is the homomorphism whose existence is guaranteed by the $\mathfrak{R}$-injective property of R. For $r \in R$, $r - i(r) = \alpha_j(r) = \alpha((r,0) + T(R^1)) = r\alpha((0,1) + T(R^1))$ $= rh$ where $h = \alpha((0,1) + T(R^1))$. Hence h is a right identity for R. By Proposition 1.9, R has an identity. Hence ${}_R\mathfrak{R}$ is the category of unital R-modules by Proposition 1.3, and all unital modules are projective since all torsion free modules are $\mathfrak{R}$-injective. Therefore, by the Artin-Wedderburn theorem, R is semi-simple Artinian.

Assume (3) and that R is a Jacobson semi-simple ring. Since R is Jacobson semi-simple, it is a subdirect sum of a set of rings $\{R_j\}_{j\in I}$ where R_j is a dense ring of linear transformations on a vector space V_j over a division ring. The projection homomorphism π_j from R to R_j is onto R_j and, if L is a closed left ideal of R_j, then it is easily seen that $\pi_j^{-1}(L)$ is a closed left ideal of R. It follows that R_j has d.c.c. on closed left ideals since R has d.c.c. on closed left ideals for each $j \in I$.

Suppose V_j is an infinite dimensional vector space for some $j \in I$. Let $\{v_1, v_2, \ldots\}$ be a basis for V_j and define:

$M_1 = \ker \alpha_1$ where $\alpha_1 : R_j \longrightarrow V_j$; $\alpha_1(r) = rv_1$

$M_2 = \ker \alpha_2$ where $\alpha_2 : M_1 \longrightarrow V_j$; $\alpha_2(r) = rv_2$

.
.
.

$M_n = \ker \alpha_n$ where $\alpha_n : M_{n-1} \longrightarrow V_j$; $\alpha_n(r) = rv_n$.

.
.
.

It is easily seen that M_{i+1} is a left ideal of R properly contained in M_i. M_{i+1} is closed in M_i since V_j is a torsion free R-module. It follows from Proposition 1.13 that M_n is a closed left ideal of R for all positive integers n. This contradicts the assumption of d.c.c. on closed left ideals of R. Hence V_j is a finite vector space for $j \in I$. Therefore R_j is a simple Artinian ring for all $j \in I$.

We assume that the family $\{R_j\}_j \in I$ has been indexed by an initial set of ordinal numbers and define $L_\circ = R$, $L_1 = \{r \in R | r(1) = 0\}$. If α has the predecessor $\alpha - 1$, then define $L_\alpha = \{r \in L_{\alpha-1} | r(\alpha) = 0\}$ and, if α is a limit ordinal, define $L_\alpha = \{r \in \bigcap_{\beta<\alpha} L_\beta | r(\alpha) = 0\}$. This clearly defines L_α for each $\alpha \in I$, and it is easily seen that L_α is a closed left ideal of R with $L_\alpha \leq L_\beta$. Since we are assuming d.c.c. on closed left ideals of R, we have that $H = \{\alpha \in I | L_\alpha$ is properly contained in L_β for all $\beta < \alpha\}$ is a finite set, i.e., $H = \{\alpha_1, \ldots, \alpha_n\}$.

We define f from R to $\oplus \sum_{i \in 1}^{n} R_{\alpha_i}$ by $f(r)\alpha_i = r(\alpha_i)$.

It is easily checked that f is a ring homomorphism onto $\oplus \sum_{i\in 1}^{n} R_{\alpha_i}$ by $f(r)\alpha_i = r(\alpha_i)$. It is easily checked that f is a ring homomorphism onto $\oplus \sum_{i=1}^{n} R_{\alpha_i}$. We now want to show that f is a monomorphism. Suppose there exists $r \in R$ such that $r \neq 0$ and $f(r) = 0$. Let β be the least ordinal such that $r(\beta) \neq 0$. β is not an element of H since $0 = f(r)\alpha_i = $ $= r(\alpha_i)$. Note that $r \in L_\sigma$ for $\sigma < \beta$ since $r(\rho) = 0$ for all $\rho < \beta$ and $r \notin L_\beta$ since $r(\beta) \neq 0$. This implies that L_β is properly contained in L_σ for all $\sigma < \beta$. However, this implies that $\beta \in H$,which is a contradiction. Hence $0 \neq r \in R$ implies $f(r) \neq 0$. Therefore, f is a ring monomorphism.

If for each $r \in R$, $r(\alpha_i) \neq 0$ implies $r(\alpha_j) \neq 0$ for some $j \neq i$, then R is a subdirect sum of $\sum_{\alpha_k \in H \ \{\alpha_i\}} \oplus R_{\alpha_k}$. Hence we can assume that for each $\alpha_i \in H$, there exists $r \in R$, such that $r(\alpha_i) \neq 0$ and $r(\alpha_j) \neq 0$ for $j \neq i$. From the above observation and noting that R_{α_i} is a simple ring with π_{α_i} restricted to R onto R_{α_i}, it is seen that f is onto $\sum_{i=1}^{n} \oplus R_{\alpha_i}$. Therefore f is an isomorphism, which proves that R is a semi-simple Artinian ring.

Comment. It is not necessary in 3.18 to assume R is Jacobson semi-simple; it is enough to assume $T(R) = 0$.

3.19 Proposition. If $R \in {}_R\mathfrak{R}$ and all torsion free R-modules are $\mathfrak{R}$-projective, then ${}_RR = {}_RN \oplus {}_RK \oplus {}_RS$ where N + K

is the Jacobson radical of R, N is the nil radical of R, and S is a semi-simple Artinian subring of R.

Proof. Assume $R \in {}_R\mathfrak{R}$ and all torsion free R-modules are $\mathfrak{R}$-projective. Let J be the Jacobson radical of R. Then R/J is a semi-simple Artinian ring by Propositions 3.17 and 3.18, and ${}_RR = {}_RJ \oplus S$ where $S \cong R/J$ by Propositions 1.34 and 3.9. Divinsky, in [3], points out that the nil radical N of R is contained in the Jacobson radical of R. Hence, by Propositions 1.14, 1.32, and 3.9, ${}_RJ = {}_RN \oplus {}_RK$. Therefore, ${}_RR = {}_RN \oplus {}_RK \oplus {}_RS$.

Comment. If R is not assumed to be an element of ${}_R\mathfrak{R}$ in the above proposition, then the conclusion holds for the ring R/T(R).

3.20 Proposition. If R is a nil semi-simple ring and all torsion free R-modules are $\mathfrak{R}$-projective, then $R = J \oplus S$ (a ring direct sum) where J is the Jacobson radical of R and S is semi-simple Artinian.

Proof. Assume R is nil semi-simple and all torsion free R-modules are $\mathfrak{R}$-projective. By Proposition 1.32, ${}_RR$ is torsion free. Hence, by Proposition 3.19, ${}_RR = {}_RJ \oplus {}_RS$ where J is the Jacobson radical of R and S is semi-simple Artinian. Since J is an ideal of R and S is a left ideal of R, we note, as in the proof of Proposition 3.15, that $JS = 0$ since R is a nil semi-simple ring. Therefore J and S are ideals of R, i.e., $R = J \oplus S$, a ring direct sum.

3.21 Proposition. If R is a ring whose only closed left ideals are 0 and R, then R is a division ring or a Jacobson

radical ring with no zero divisors.

Proof. Assume the only closed left ideals of R are 0 and R. Then T(R) = 0 since 0 is closed. Hence R does not annihilate any elements of R, which implies that R does not have any zero divisors since a zero divisor would determine an R homomorphism from R into R whose kernel would be a proper closed left ideal of R. Since the Jacobson radical or R is a closed left ideal, we have that R is Jacobson semi-simple or a radical ring. If R is Jacobson semi-simple, then it follows from 3.18 and the comment following 3.19 that R is a division ring since R has d.c.c. on closed left ideals.

Using the relationships between the torsion theories $({}_R\mathfrak{T}, {}_R\mathfrak{R})$, $(\mathfrak{T}_2, \mathfrak{R}_2)$, and $(\mathfrak{T}_3, \mathfrak{R}_3)$ established in Section II, we now relate the results of this section to the torsion theories $(\mathfrak{T}_2, \mathfrak{R}_2)$ and $(\mathfrak{T}_3, \mathfrak{R}_3)$.

3.22 Definition. For $(\mathfrak{T}, \mathfrak{R})$ a torsion theory on ${}_R\mathfrak{M}$, an $\mathfrak{R}$-torsion free R-module M will be called $\mathfrak{R}$-injective if it is injective in the full subcategory $\mathfrak{R}$ of ${}_R\mathfrak{M}$, and $\mathfrak{R}$-projective if it is projective in the full subcategory $\mathfrak{R}$ with respect to the onto epimorphisms in $\mathfrak{R}$, as in 1.48.

3.23 Proposition. If $(\mathfrak{T}, \mathfrak{R})$ is a torsion theory on ${}_R\mathfrak{M}$ closed under taking submodules, then all $\mathfrak{R}$-torsion free modules are $\mathfrak{R}$-injective if and only if all $\mathfrak{R}$-torsion free modules are injective, i.e., injective in the category ${}_R\mathfrak{M}$.

Proof. Assume $(\mathfrak{T}, \mathfrak{R})$ is a torsion theory on ${}_R\mathfrak{M}$ closed under taking submodules and all $\mathfrak{R}$-torsion free modules are $\mathfrak{R}$-injective. For a submodule L of M, t(L) is

contained in $t(M)$ by 2.3(7) where, for any R-module N, $t(N)$ is the sum of the $\mathfrak{T}$-torsion submodules of N. Hence $t(L)$ is contained in $t(M) \cap L$. Note that $t(M) \cap L \in \mathfrak{T}$ since $t(M)$ is torsion and $\mathfrak{T}$ is closed under taking submodules. Therefore, $t(L) = t(M) \cap L$.

Suppose Q is $\mathfrak{R}$-injective and consider the following diagram of R-homomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\alpha"'] & H \\
 & Q &
\end{tikzcd}$$

Note that it gives rise to the following diagram of homomorphisms since α

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "j"] \arrow[drr] \arrow[dd, "\alpha"'] & H & \\
0 \arrow[rrr] & & & M/t(M) \arrow[r, "i"] \arrow[dll] & H/t(H) \arrow[dlll, dashed, "\beta"] \\
 & Q & & &
\end{tikzcd}$$

restricted to $t(M) = 0$ by 2.3(5) and i is a monomorphism since $j(t(M)) = t(j(M)) = t(H) \cap j(M) = j(t(M))$ from the above argument. β is the R-homomorphism guaranteed by the $\mathfrak{R}$-injectivity of Q. Hence it is easily seen that Q is injective in ${}_R\mathfrak{M}$.

The converse is immediate.

Alin and Dickson, in [1], proved that if R is a ring with identity, then all ${}_RM \in \mathfrak{R}_3$ are injective if and only if

$R = T \oplus S$ where T has an essential singular ideal and S is semi-simple Artinian. The following proposition is very similar to Alin and Dickson's result.

3.24 Proposition. If the image of every essential left ideal of R is an essential left ideal in the ring $R/Z_2(R)$, then all ${}_RM \in \mathfrak{R}_3$ are injective if and only if $R = K \oplus S$ where $Z_2(K) = K$ and S is a semi-simple Artinian ring.

Proof. Assume the image of every essential left ideal of R is an essential left ideal in the ring $R/Z_2(R)$ and all ${}_RM \in \mathfrak{R}_3$ are injective. From 2.3(7) and 2.13, we note that $T_3(R) = Z_2(R)$. It is easily seen that $T_3(R)M = 0$ for $M \in \mathfrak{R}_3$. We will write $R/Z_2(R)$ as $\bar{R}$. It follows from the assumption on R that ${}_RM$ is $\mathfrak{R}_3$-torsion free if and only if ${}_{\bar{R}}M$ is $\mathfrak{R}_3$-torsion free where we define $rm \equiv (r+T_3(R))m$. Hence all $\mathfrak{R}_3$-torsion free $\bar{R}$-modules are $\mathfrak{R}_3$-injective. This implies that $\bar{R}$ does not contain an essential left ideal since submodules of ${}_{\bar{R}}\bar{R}$ are $\mathfrak{R}_3$-torsion free. Hence $\bar{R}$ is a full ring which implies that ${}_{\bar{R}}M \in \mathfrak{R}_3$ if and only if ${}_{\bar{R}}M \in {}_{\bar{R}}\mathfrak{R}$. This implies that all torsion free $\bar{R}$-modules are injective. Noting that $T(\bar{R}) = 0$ since $Z(\bar{R}) = 0$, we have $\bar{R}$ is a semi-simple Artinian ring by 3.18. Hence

$${}_R\bar{R} = \sum_{i=1}^{n} \oplus\ {}_{\bar{R}}\bar{R}\bar{e}_i = \sum_{i=1}^{n} \oplus\ {}_R\bar{R}\bar{e}_i, \text{ where}$$

$\bar{e}_i\bar{e}_j = S_{ij}$ amd ${}_R\bar{R}\bar{e}_i$ is a simple R-module for $i = 1, \ldots, n$. Consider the sequence of homomorphisms

$${}_R\bar{R} = \xrightarrow{\alpha} \sum_{i=1}^{n} \oplus\ {}_R\bar{R}\bar{e}_i \xrightarrow{\beta_j} {}_R\bar{R}\bar{e}_j, \text{ where}$$

β_j is the projection homomorphism and α is the canonical homomorphism from R onto $R/T_3(R)$. Note that $\bar{e}_j \in {}_R R\bar{e}_j$ and $\bar{e}_j\bar{e}_j = \bar{e}_j = \alpha(s_j) = \alpha(s_j)\alpha(s_j) = \alpha(s_j s_j) = s_j\alpha(s_j)$, where $s_j \in \alpha^{-1}(\bar{e}_j)$. Ker $\alpha\beta_j$ is not an essential left ideal of R since $R\bar{e}_j$ is $\mathfrak{R}_3$-torsion free, being a submodule of $\bar{R}$. Hence there exists H_j a left ideal of R such that $H_j \cap \ker \alpha\beta_j = 0$, which implies that α restricted to ${}_R H_j$ is an isomorphism onto ${}_R R\bar{e}_j$ for $j = 1, \ldots, n$. Suppose $H_j = H_i$ where $i \neq j$. Then there exist r_j, $r_i \in H_j$ such that $\alpha(r_j) = \bar{e}_j$ and $\alpha(r_i) = \bar{e}_i$. However, $\alpha(r_j r_i) = \alpha(r_j)\alpha(r_i) = \bar{e}_j\bar{e}_i = 0$, which implies that $r_j r_i = 0$. This implies that ker σ_i is a proper submodule of ${}_R H_j$ where $\sigma_i(r) = rr_i$ since $r_i r_i \neq 0$ and $r_j r_i = 0$. This is impossible since H_j is a simple module. Hence $H_j \neq H_i$ if $i \neq j$. We now note that, if $h_1 + \ldots + h_n = 0$, where $h_j \in H_j$ for $j = 1, \ldots, n$, then $\alpha(h_1)\bar{e}_j + \ldots + \alpha(h_n)\bar{e}_j = 0$, and this implies that $\alpha(h_j)\bar{e}_j = 0$, from which it follows that $h_j = 0$. Hence it is easily seen that ${}_R R = {}_R H \oplus {}_R Z_2(R)$ where α restricted to $H = \sum_{j=1}^{n} H_j$ an isomorphism onto $\bar{R}$. Note that $\mathfrak{R}_3$ is closed under homomorphic images since all $\mathfrak{R}_3$-torsion free modules are injective. Therefore, since ${}_R H$ is $\mathfrak{T}_3$-torsion, we have that $R = H \oplus Z_2(R)$ is a ring direct sum since right multiplications in R are left R-homomorphisms.

Conversely, assume that $R = K \oplus S$ where $Z_2(K) = K$ and S is semi-simple Artinian. Clearly, $Z_2(R) = K$ and ${}_R M$ is $\mathfrak{R}_3$-torsion free if and only if ${}_{R/T_3(R)}M$ is $\mathfrak{R}_3$-torsion free. Since $R/T_3(R)$ is a ring isomorphic to S and S is a

full ring, we have $\mathfrak{R}_3 = {}_S\mathfrak{R} = {}_S\mathfrak{M}$ where ${}_S\mathfrak{M}$ is the category of unital S-modules. Hence all $\mathfrak{R}_3$-torsion free modules are $\mathfrak{R}_3$-injective.

3.25 Proposition. If $Z_2(R) = 0$ and all $\mathfrak{R}_3$-torsion free modules are injective, then R is semi-simple Artinian.

Proof. This is a corollary of the previous proposition. However, it also follows very quickly from Proposition 3.18.

3.26 Proposition. If $T(R) = 0$, then all $\mathfrak{R}_2$-torsion free modules are $\mathfrak{R}_2$-injective if and only if R is semi-simple Artinian.

Proof. Assume $T(R) = 0$ and all $\mathfrak{R}_2$-torsion free modules are $\mathfrak{R}_2$-injective. From the definition of $\mathfrak{J}_2$ it immediately follows that ${}_RR$ is $\mathfrak{R}_2$-torsion free. Hence R contains no essential left ideals since any left ideal of R is $\mathfrak{R}_2$-injective. By 2.26 and 2.15, ${}_R\mathfrak{R} = \mathfrak{R}_2$ since R is a full ring and $T(R) = 0$. This implies that all torsion free modules are $\mathfrak{R}$-injective. By 3.18, R is semi-simple Artinian. The converse follows easily from the theory of modules for semi-simple Artinian rings. It also can be seen to follow easily from Propositions 2.26, 2.15, and 1.3.

3.27 Proposition. If R is a torsion free full ring and all $(\mathfrak{R}_2)\mathfrak{R}_3$-torsion free modules are $(\mathfrak{R}_2)\mathfrak{R}_3$-projective, then

(a) ${}_RR = {}_RJ \oplus {}_RS$ where J is the Jacobson radical of R and S is a semi-simple Artinian subring of R,

(b) $R = J \oplus S$ (a ring direct sum) if R is nil semi-simple,

(c) $R = S$ if R is Jacobson semi-simple (J and S as in part (a)).

Proof. The proof for the $\mathfrak{R}_3$ case follows from 2.26, 3.18, 3.19 and 3.20.

SECTION IV

A CLOSURE OPERATOR ON SUBMODULES AND A CLASS OF RINGS WHICH SATISFIES A CERTAIN CLOSURE PROPERTY

In this section the properties of a closure operator on the submodules of an R-module M are discussed. It is shown that the lattice of closed submodules of an R-module is a complete modular lattice which is complemented if and only if R is a full ring. The properties of full rings are also investigated.

We will first consider the closure of L in M. To do this, we generalize the definition of closure as given in 2.21.

Let L be a submodule of M and $(\mathfrak{J},\mathfrak{R})$, a torsion theory on ${}_R\mathfrak{M}$, with torsion functor t as discussed in 2.3(6) and (7). L, a submodule of M, will be called <u>closed in M</u> with respect to $(\mathfrak{J},\mathfrak{R})$ if $M/L \in \mathfrak{R}$. We define the <u>closure of L in M with respect to $(\mathfrak{J},\mathfrak{R})$</u> to be the intersection of all closed submodules in M with respect to $(\mathfrak{J},\mathfrak{R})$ which contain L. It will be written $cl_{\mathfrak{J}}L_M$. We shall omit the $\mathfrak{J}$ when discussing ${}_R\mathfrak{J}$ and the M when it will not lead to confusion.

4.1 <u>Proposition</u>. If L is a submodule of M and $(\mathfrak{J},\mathfrak{R})$ is a torsion theory on ${}_R\mathfrak{M}$, then $M/cl_{\mathfrak{J}}L \in \mathfrak{R}$.

Proof. Let $\aleph$ be the set of all closed submodules of M which contain L. Define an R-module homomorphism from M onto $\oplus \sum_{L_\alpha \in \aleph} M/L_\alpha$ where $g(m)\alpha_\circ = m + L_{\alpha_\circ}$. Clearly $\ker g = \bigcap_{L_\alpha \in \aleph} L_\alpha$ and $\oplus \sum M/L_{\alpha_\circ} \in \mathfrak{R}$. It follows that $M/\mathrm{cl}\, L \in \mathfrak{R}$.

4.2 Proposition. If L is a submodule of M, then $\sigma^{-1}(t(M/L)) = \mathrm{cl}_{\mathfrak{J}} L_M$ where σ is the canonical f homomorphism from M onto M/L and t is the torsion functor for $(\mathfrak{J}, \mathfrak{R})$.

Proof. $\sigma^{-1}(t(M/L))$ is closed in M since $M/L/t(M/L) \in \mathfrak{R}$. Hence $\mathrm{cl}_{\mathfrak{J}} L$ is contained in $\sigma^{-1}(t(M/L))$. Suppose L' is a closed submodule of M containing L. Then $\ker \beta$ is closed in M/L where β is the canonical homomorphism from M/L onto M/L'. This implies, by 2.3(6), that $r(M/L)$ is contained in $\ker \beta$. It follows that L' is contained in $\sigma^{-1}(t(M/L))$. Hence $\sigma^{-1}(t(M/L))$ is contained in cl L. Therefore $\mathrm{cl}_{\mathfrak{J}} L = \sigma^{-1}(t(M/L))$.

4.3 Proposition. If $(\mathfrak{J}', \mathfrak{R}')$ and $(\mathfrak{J}'', \mathfrak{R}'')$ are torsion theories on ${}_R\mathfrak{M}$ with $\mathfrak{J}'$ contained in $\mathfrak{J}''$, then, for L a submodule of M, $\mathrm{cl}_{\mathfrak{J}'}(L)_M$ is contained in $\mathrm{cl}_{\mathfrak{J}''}(L)_M$ and $\mathrm{cl}_{\mathfrak{J}'}(\mathrm{cl}_{\mathfrak{J}''}(L)_M)_M = \mathrm{cl}_{\mathfrak{J}''}(L)_M$.

Proof. The proof follows from 2.3(6) and 4.2.

Let $L \subset M$ be submodules of a module N, then we have

4.4 Definition. For a submodule L of M we define $L_\circ = L$, $L_1 = \{m \in M \mid Rm \subseteq L\}$ and for α, an ordinal number with predecessor $\alpha - 1$, we define $L_\alpha = \{m \in M \mid Rm \subseteq L_{\alpha-1}\}$; for α, a limit ordinal, we define $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$.

4.5 Proposition. If L is a submodule of M, then $\mathrm{cl}\ {}_R\mathfrak{J}L_M = L_\sigma$ for some ordinal σ.

Proof. This follows from 1.27 and 4.2.

4.6 Proposition. Let $L \subset M$ be submodule of a module N, then

(a) $\mathrm{cl}\ L_M = \mathrm{cl}\ L_N \cap M$.

(b) $M \in {}_R\mathfrak{R}$ implies that $\mathrm{cl}\ L_M$ is an essential extension of L.

(c) M a closed submodule of N implies $\mathrm{cl}\ L_M = \mathrm{cl}\ L_N$.

(d) L is closed in M if and only if $L = \ker \alpha$ where α is a R-homomorphism from M into a torsion free module.

Proof. (a) By 4.2 and 2.5, we have the following equalities where α is the canonical homomorphism from N to N/L and $\alpha_\circ = \alpha|M$: $\mathrm{cl}\ L_M = \alpha_\circ^{-1}(T(M/L)) = \alpha_\circ^{-1}(T(N/L)\cap M/L)$ $= \alpha^{-1}(T(N/L)\cap\alpha_\circ^{-1}(M/L) = \alpha^{-1}(T(N/L))\cap M = \mathrm{cl}\ L_N \cap M$.

(b) L a submodule of a torsion free module M implies that E(M) is a torsion free module and E(L) can be taken to be a submodule of E(M) where, for N an R-module, E(N) is the injective hull of N. E(L) is closed in E(M) since it is a direct summand of E(M). Hence $E(L) \cap M$ is closed in M, which implies that $\mathrm{cl}\ L_M$ is contained in $E(L) \cap M$. Therefore cl L is an essential extension of L since E(L) is an essential extension of L.

Parts (c) and (d) are easily seen.

4.7 Proposition. If L and H are submodules of M, then $\mathrm{cl}\ L_M \cap \mathrm{cl}\ H_M = \mathrm{cl}(L \cap H)_M$ and $\mathrm{cl}\ L_M + \mathrm{cl}\ H_M \subseteq \mathrm{cl}(L + H)_M$.

Proof. The only part of this proposition which is not immediate is $\mathrm{cl}\ L_M \cap \mathrm{cl}\ H_M \subseteq \mathrm{cl}(L \cap H)_M$. Using the description of $\mathrm{cl}\ L_M$ given in 4.5, we will show $\mathrm{cl}\ L_M \cap H \subseteq \mathrm{cl}(L \cap H)_M$. Suppose $m \in L_1 \cap H$; then $Rm \subseteq L$ and $m \in H$ which implies that $Rm \subseteq L \cap H$. This implies that $m \in \mathrm{cl}(L \cap H)_M$. Assume $L_\alpha \cap H \subseteq \mathrm{cl}(L \cap H)$ for all $\alpha < \beta$. Let $m \in L_\beta \cap H$. Then if β has a predecessor, $\beta - 1$, then $Rm \subseteq L_{\beta-1} \cap H \subseteq \mathrm{cl}(L \cap H)$ which implies $m \in \mathrm{cl}(L \cap H)$; if β is a limit ordinal, then $m \in L_\alpha \cap H$ for some $\alpha < \beta$ which implies that $m \in \mathrm{cl}(L \cap H)$. It follows, by 4.5, that $\mathrm{cl}\ L_M \cap H \subseteq \mathrm{cl}(L \cap H)_M$. Similarly, it follows that $\mathrm{cl}\ L_M \cap \mathrm{cl}\ H_M \subseteq \mathrm{cl}(\mathrm{cl}\ L_M \cap H)_M \subseteq \mathrm{cl}(L \cap H)_M$. This proves the proposition.

Examples. Let $R = 2Z$ (Z the integers) and consider the family of ideals of $\{L_k\}_{k\in N}$ (N the counting numbers), where $L_k = 2^k Z$ for $k \in N$. It is easily observed that $(\mathrm{cl}\ L_k)_R = R$ for all $k \in N$. Hence $\mathrm{cl}(\bigcap_{k\in N} L_k)_R \neq \bigcap_{k\in N} (\mathrm{cl}\ L_k)_R$. Therefore the closure of the intersection of an infinite family of left ideals is not necessarily equal to the intersection of the closure of the same family of left ideals.

Let V be a countably infinite dimensional vector space with basis $\{v_j\}_{j\in N'}$, and let R be the subring of the ring of linear transformations which contains all linear transformations of finite rank and the transformation α where $\alpha(V_{3j}) = V_{3j+1}$, $\alpha(V_{3j+1}) = 0$, and $\alpha(V_{3j+2}) = 0$ for $j \in N'$ ($N' = N \cup \{0\}$). Clearly, $r\alpha$ or αr is of finite rank for $r \in R$. Let $I_1 = \{r \in R \mid r \neq \alpha$ and $r(v_{2j}) = 0$ for all

$j \in N'\}$. One easily notes that I_1 and I_2 are closed left ideals of R and that cl I_1 + cl $I_2 \subsetneq cl(I_1 + I_2) = R$. Therefore the sum of the closure of left ideals is not necessarily equal of the closure of the sum of left ideals.

4.8 Definition. An element L of $_R\mathfrak{R}$ is said to be pure if L is closed in any torsion free module containing it.

The following proposition is noted by Alin and Dickson in [1].

4.9 Proposition. $L \in \mathfrak{R}$ is pure if and only if E(L)/L is torsion free.

Proof. Assume L is a closed submodule of E(L) and $E(L) \in \mathfrak{R}$. If L is contained in a torsion free module M, then E(L) is clearly closed in E(M). (E(L) can be considered a submodule of E(M)), which implies that $E(L) \cap M$ is a closed submodule of M. Hence L is closed in M.

The converse is immediate.

4.10 Definition. For L and H closed submodules of M, we define $L \vee H = cl(L + H)_M$ and $L \wedge H = cl\ L_M \cap cl\ H_M$.

4.11 Proposition. The set of closed submodules of an R-module M, with binary operations $\vee$ and $\wedge$ defined on it as above, forms a complete modular lattice.

Proof. The only part of the statement which is not immediate is that it is a modular lattice.

Suppose A, B, and C are closed submodules of $_RM$ with $B \subseteq C$ and $A \cap C = 0$. Claim $(A \vee B) \wedge C = B$. Note $A + B \cap C = B$. Assume $(A + B)_\alpha \cap C = B$ if $\alpha < \beta$, using the notation of 4.4. If β is a limit ordinal, then clearly

$(A + B)_\beta \cap C = B$. If β has $\beta - 1$ as a predecessor and $m \in (A + B)_\beta \cap C$, then $Rm \subseteq (A + B)_{\beta-1}$ and $m \in C$. This implies Rm is contained in B since $(A + B)_{\beta-1} \cap C = B$ by the inductive assumptions, which implies that $m \in B$ since B is a closed submodule of M. Hence $(A + B)_\beta \cap C = B$. From Proposition 4.5, $cl(A + B)_M \cap C = (A \vee B) \wedge C = B$. It follows from [9, p. 167], that the lattice of closed submodules is modular.

4.12 Proposition. A submodule H of M is closed in M with respect to $\mathfrak{R}_3$ if and only if for $m \in M \setminus H$ there exists $r \in R$ such that $Rr + r \cap (m + H)_L = 0$ where $(m + H) \in M/H$.

4.13 Proposition. The following are equivalent: (L is a submodule of an arbitrary R-module M.)

(a) $cl_{\mathfrak{R}_3} L_M = cl_{R\mathfrak{R}} L_M$;

(b) R is a full ring;

(c) The lattice of closed submodules of an R-module is complemented.

Proof. Noting that for an arbitrary torsion theory $(\mathfrak{T}, \mathfrak{R})$ on ${}_R\mathfrak{M}$, $t(M) = cl_{\mathfrak{R}}\{0\}_M$, where t is the torsion functor, we have that (a) implies ${}_R\mathfrak{T} = \mathfrak{T}_3$. Hence by 2.15 and 2.25 (a) implies (b). From 2.25, one easily sees that (b) implies (a).

Assume (b) and suppose that H is a closed submodule of N. Let $n \in N \setminus H$. Since (b) and (c) are equivalent, H is closed in N with respect to $\mathfrak{R}_3$. Hence there exists $r \in R$ such that $rn \notin H$ and $Rr \cap (n + H)_L = 0$. This implies that $Rrn \cap H = 0$ and $Rrn \neq 0$ since $r(n + H) \neq 0$ and $(n + H)$ is

an element of the torsion free module N/H. It follows that there exists a nonzero submodule K of N such that $H \cap K = 0$, and any submodule of N that properly contains K intersects H nontrivially. It is easily seen that K is closed in N and that $K + H$ is an essential submodule of N. Hence $\mathrm{cl}(H + K)_N = N$. Therefore, the lattice of closed submodules of a module is complemented.

That (c) $\Rightarrow$ (b) is immediate. This proves the proposition.

4.14 Proposition. If $T(R) = 0$, the following are equivalent:

(a) R is a full ring;

(b) For a torsion free R-module M and a submodule L of M, $\mathrm{cl}\, L_M = E(L) \cap M$:

(c) For N a torsion free R-module N and a submodule H of N, $\mathrm{cl}\, H_N = \{n \in N \mid rn \neq 0 \text{ implies } Rrn \cap H \neq 0 \text{ for all } r \in R\}$.

Proof. Assume (a) and that M is a torsion free R-module with L a submodule of M. Take $E(L)$ to be a submodule of $E(M)$. Then $\mathrm{cl}\, L_{E(M)}$ is contained in $E(L)$ by 4.6(c). From 4.6(a), it follows that $\mathrm{cl}\, L_M$ is contained in $E(L) \cap M$.

Let $m \in E(L) \cap M$ and suppose that $rm \notin L$. Then $Rrm \neq 0$ since M is torsion free. This implies that $Rrm \cap L \neq 0$ since $E(L)$ is an essential extension of L. Hence $m \in \mathrm{cl}\, L_M$ by 4.12 and 4.13. Therefore $\mathrm{cl}\, L_M = E(L) \cap M$ where $E(L)$ is contained in $E(M)$.

Assume (b) and suppose K is an essential left ideal of R. Then, if E(K) is taken as a submodule of E(R), $E(K) \cap R = R$ since R is an essential extension of K. Hence (b) implies cl $K_R = R$, i.e., R is a full ring. Therefore, (a) and (b) are equivalent, assuming $T(R) = 0$.

Assume (a) and let N be a torsion free R-module with H a submodule. Define $D = \{n \in N | rn \neq 0$ implies $Rrn \cap H \neq 0$ for all $r \in R\}$. Since cl H_N = cl $_{\mathcal{J}_3}H_N$, it follows by 4.14 that D is contained in cl H_N. It is easily seen by transfinite induction that cl H_N is contained in D by 4.4 and 4.5. Hence cl $H_N = D$.

It is obvious that (c) implies (a). This completes the proof.

4.15 Proposition. If R is a full ring, then the lattice of closed submodules of an R-module has a.c.c. on closed submodules if and only if it has d.c.c. on closed submodules.

Proof. Assume R is a full ring and M is an R-module with d.c.c. on closed submodules. Suppose C_1 and C_2 are proper closed submodules in M with C_1 properly contained in C_2. There exists a closed submodule B_1 in M such that $cl(B_1 + C_1) = M$ and $B_1 \cap C_1 = 0$ by Proposition 4.13.

If $C_2 \wedge B_1$ is an essential submodule of B_1, then C_2 contains B_1 since $C_2 \cap B_1$ is a closed submodule of B_1. Hence $C_2 \cap B_1$ is a closed submodule of B_1. We note that $C_2 \cap B_1$ is not an essential submodule of B_1 since $cl(C_2 + B_1) = M$ and C_1 is properly contained in M. Since the lattice of closed submodules of M is modular, $C_2 \cap B_1 \neq 0$. Let B_2 be

a submodule of B_1 maximal with respect to the property that $B_2 \cap C_2 = 0$. It is easily checked that B_2 is a closed submodule of B_1. Therefore B_2 is a closed submodule of M properly contained in B_1 such that $B_2 \vee C_2 = M$ and $B_2 \wedge C_2 = 0$. It follows that M has a.c.c. on closed submodules.

The proof that a.c.c. on closed submodules of an R-module M implies d.c.c., assuming R is a full ring, is similar to the above proof.

4.16 Definition. N and H submodules of M are said to be related if $N \cap X = 0$ if and only if $H \cap X = 0$ for any submodule X of M.

4.17 Proposition. If N and H are submodules of M, then:

(1) $\mathrm{cl}\ \Re_3(N)_M \sim N + \mathrm{cl}\Re_3(0)_M$.

(2) $P \sim N \Rightarrow P \subseteq \mathrm{cl}\ \Re_3 N_M$.

This is proven by Goldie. (See [6, p. 169].)

4.18 Recall that a module M is said to be finite dimensional if it does not contain an infinite direct sum of nonzero submodules.

4.19 Recall that a submodule U of M is said to be uniform if $U \neq 0$ and every pair of nonzero submodules of U has nonzero intersections. (See [6, Chapter 3].)

4.20 Definition. A nonzero submodule L of M will be called a minimal closed submodule if it is closed and does not contain a proper closed submodule.

4.21 Proposition. If M is a finite dimensional torsion free R-module, then every nonzero submodule contains a uniform submodule.

Proof. See [6, Chapter 3].

4.22 Proposition. If U is a uniform submodule of M and $M \in {}_R\mathfrak{R}$, then $\mathrm{cl}\ U_M$ is a uniform submodule of M.

Proof. This follows directly from 4.6(b).

4.23 Proposition. A minimal closed submodule of a torsion free module is a uniform submodule.

Proof. The proof follows immediately from 4.6(b).

4.24 Proposition. If R is a full ring, then L is a minimal closed submodule of a torsion free module M if and only if L is a maximal uniform submodule of M, i.e., there does not exist a uniform submodule of M which properly contains L.

4.25 Proposition. If M is a finite dimensional torsion free R-module, then there exists an integer $n \geq 0$ such that:

(1) any direct sum of uniform submodules of M having maximal length has n terms;

(2) every direct sum of nonzero submodules of M has at most n terms;

(3) a submodule N of M is essential if and only if it contains a direct sum of n uniform submodules.

Proof. See [6, Chapter 3].

4.26 Proposition. If R is a full ring and M is a torsion free module of dimension n, then $M = \mathrm{cl}(U_1 \oplus \ldots \oplus U_n)$ where U_j is a minimal closed submodule of M for $j = 1, \ldots, n$.

4.27 Proposition. If R is a torsion free full ring and ${}_RR$ has dimension n, then ${}_RR = \mathrm{cl}(U_1 \oplus \ldots \oplus U_n)$ where U_j is a minimal closed submodule of R and, as a subring of R, U_j is a zero ring or has no divisors of zero for $j = 1, \ldots, n$.

4.28 Proposition. The ring direct sum of full rings is a full ring.

4.29 Proposition. The homomorphic image of a full ring is a full ring.

4.30 Proposition. If R is a dense ring of linear transformations on a vector space V over a division ring D with Soc R $\neq$ 0, then R is a full ring if and only if ${}_R R$ = cl Soc R.

4.31 Proposition. If R is a full ring, then R has d.c.c. on closed left ideals if and only if R is finite dimensional.

Comment. If R is not assumed to be full in 4.31, then the proposition does not hold since R = 2Z is a finite dimensional ring which does not have d.c.c. on closed left ideals.

SECTION V

A LIST OF OPEN QUESTIONS

1. We noted in Section I that $T(R)M \subseteq T(M)$. Under what conditions does equality hold? (W. E. Clark)

2. If R has an identity, then it is easily seen that $_RR$ is isomorphic to $_R(R^1/T(R^1))$. Does the converse of this theorem hold?

3. If R is a torsion free Jacobson radical ring, is it possible that all torsion free modules are $\mathfrak{R}$-projective? (W. E. Clark)

4. Let R be the subring of the 2×2 matrix ring with rational entries such that the elements of R are of the form $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$. Are all torsion free R-modules $\mathfrak{R}$-projective? (W. E. Clark)

5. If R is Jacobson semi-simple and all $(\mathfrak{R}_2)\mathfrak{R}_3$-torsion free R-modules are $(\mathfrak{R}_2)\mathfrak{R}_3$-projective, is R semi-simple Artinian?

6. What conditions on R are necessary to guarantee that $\bigcap_{a\in A} (\overline{L_a})_M = (\overline{\bigcap_{a\in A} L_a})_M$ where $\{L_a\}_{a\in A}$ is a set of submodules of a torsion free module M?

7. Give a ring-theoretical or module-theoretical characterization of rings of the form $\bigoplus_{i \in I} S_i$ where I is possibly infinite and S_i is a semi-simple Artinian ring for each i. (W. E. Clark)

8. If R has no identity, is it true that the categories ${}_R\mathfrak{M}$ and ${}_{R_n}\mathfrak{M}$, where R_n is the n × n matrix ring with entries from R, are equivalent? This holds for unital modules over rings with identity. However, $(R^1)_n \not\cong (R_n)^1$, so this is apparently not helpful. (W. E. Clark)

9. If R is nil semi-simple and has d.c.c. on closed left ideals, does R = J ⊕ S where J is the Jacobson radical and S is Artinian semi-simple?

10. Is it possible to characterize those rings R which are of the form I^1 for some ideal I in R by considering the properties of the torsion theory $({}_I\mathfrak{T}, {}_I\mathfrak{R})$? (W. E. Clark)

11. Is it possible to find necessary and sufficient conditions on R such that $\mathrm{cl}\ L_M + \mathrm{cl}\ H_M = \mathrm{cl}(L + H)_M$ for all R-modules M?

BIBLIOGRAPHY

[1] J. S. Alin and S. E. Dickson. "Goldie's Torsion Theory and its Derived Functor." To be published.

[2] S. E. Dickson. "A Torsion Theory for Abelian Categories," Trans. Am. Math. Soc. 121 (1966), 223-235.

[3] N. J. Divinsky. Rings and Radicals, University of Toronto Press (1965).

[4] E. Gentile. "Singular submodule and injective hull," Indag. Math. 24 (1962), 426-433.

[5] A. W. Goldie. "Torsion free modules and rings," J. Algebra 1 (1964), 268-287.

[6] A. W. Goldie. "Rings with Maximum Condition," Yale University Notes, 1964.

[7] N. Jacobson. "Structure of rings," Am. Math. Colloq. Pub., Vol. 37 (1956).

[8] J. P. Jans. "Some aspects of torsion," Pac. Jour. Math., Vol. 15, No. 4, (1965).

[9] A. G. Kurosh. Lectures on General Algebra, Chelsea Pub. Co., New York (1963).

[10] B. Mitchell. Theory of Categories, Academic Press, New York and London (1965).

BIOGRAPHICAL SKETCH

John Michael Kellett was born December 19, 1935, in Milford, Massachusetts. He was graduated from St. Mary's High School, Milford, in June, 1953, and received his Bachelor of Science degree from Worcester State College in June, 1957. He taught high school mathematics for a year and a half before entering the United States Navy. While serving as a naval officer, he attended the University of New Mexico as a part-time student.

In 1965, the author received his Master of Science degree in mathematics from Rutgers - The State University. In September of that year, he came to the University of Florida where he is presently an interim instructor and is working toward the degree of Doctor of Philosophy.

Mr. Kellett is married to the former Christine Hunter of Gainesville. They have four children, William Hunter, John Michael, Jr., Jane Therese, and Paul Cyril.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1968

E Ruffin Jones
Dean, College of Arts & Sciences

Dean, The Graduate School

Supervisory Committee:

W. Edwin Clark
Chairman

[illegible]

J. O. Moore

R H Bowers